THE UNIVERSITY of EDINBURGH

This thesis has been submitted in fulfilment of the requirements for a postgraduate degree (e.g. PhD, MPhil, DClinPsychol) at the University of Edinburgh. Please note the following terms and conditions of use:

- This work is protected by copyright and other intellectual property rights, which are retained by the thesis author, unless otherwise stated.
- A copy can be downloaded for personal non-commercial research or study, without prior permission or charge.
- This thesis cannot be reproduced or quoted extensively from without first obtaining permission in writing from the author.
- The content must not be changed in any way or sold commercially in any format or medium without the formal permission of the author.
- When referring to this work, full bibliographic details including the author, title, awarding institution and date of the thesis must be given.

Combinatorics of Gaudin systems: cactus groups and the RSK algorithm

Noah White

Doctor of Philosophy
University of Edinburgh
2016

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. The work has not been submitted for any other degree or professional qualification.

(*Noah White*)
September 7, 2016

Abstract

This thesis explores connections between the Gaudin Hamiltonians in type A and the combinatorics of tableaux. The cactus group acts on standard tableaux via the Schützenberger involution. We show in this thesis that the action of the cactus group on standard tableaux can be recovered as a monodromy action of the cactus group on the simultaneous spectrum of the Gaudin Hamiltonians. More precisely, we consider the action of the Bethe algebra, which contains the Gaudin Hamiltonians, on the multiplicity space of a tensor product of irreducible $\mathfrak{gl}_r$-modules. The spectrum of this algebra forms a flat and finite family over $M_{0,n+1}(\mathbb{C})$. We use work of Mukhin, Tarasov and Varchenko, who link this spectrum to certain Schubert intersections, and work of Speyer, who extends these Schubert intersections to a flat and finite map over the entire moduli space of stable curves $\overline{M}_{0,n+1}(\mathbb{C})$. We show the monodromy over the real points $\overline{M}_{0,n+1}(\mathbb{R})$ can be identified with the action of the cactus group on a tensor product of irreducible $\mathfrak{gl}_r$-crystals. Furthermore we show this identification is canonical with respect to natural labelling sets on both sides.

Lay summary

This thesis is about the *representation theory* of the *general linear group*. Let us first explain these terms; a *group* is a mathematical object which encodes a set of symmetries. For example, we are all familiar with the idea of three dimensional space, most of us need to navigate it every day. As mathematicians we do not think of the number three as being special, so we are happy to speak about n-dimensional space, for some number n. One way to imagine this is to think how you would give someone precise instructions to build a three dimensional box; you would need three numbers, length, depth and height. In n-dimensions we would need to give n-numbers to our carpenter to build an n-dimensional box. The general linear group is the collection of all transformations of n-dimensional space that fix a specified point and send straight lines to straight lines. Examples are rotations about a point in space, or reflections, or even just the transformation which does nothing! The important properties of this collection are that given a transformation, we can always find another to undo the first (e.g. rotate in the opposite direction, or reflect again), and given two transformations doing them one after another still preserves straight lines and the fixed point.

The general linear group is an example of a group. Often groups can be quite complicated and the best way to study them is to see if they turn up as the symmetries of any particular spaces. In fact, our definition of the general linear group says that it is the symmetries of n-dimensional space. This is the *fundamental representation*, but there are many more representations and some have many dimensions indeed! Nonetheless, this turns out to be a very effective way to study our group, in particular because we are able to describe explicitly all of the representations using *combinatorial* objects.

Combinatorics can be thought of as the mathematical study of counting. An example of a combinatorial problem is: *how many ways are there of arranging three red balls and two green balls in a line?* (Answer: 10). Another combinatorial problem, that turns out to need some geometrical thinking to solve: *given four randomly chosen straight lines in three dimensional space, how many straight lines intersect all four?* (Answer: 2). If we choose the right combinatorial objects, then they can be very useful for encoding difficult properties of other mathematical structures.

For representations of the general linear group, the correct combinatorial objects to

consider are *tableaux*. Tableaux are arrangements of boxes, left justified and each row has an equal number, or fewer boxes than the row above it. We are required to number the boxes so that each row and column is increasing. An example of a tableau is

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}.$$

There is an important operation one can perform on tableaux called the *Schützenberger involution*. This operation takes a tableau and transforms it into another. First we empty the top left box of the tableau, and remember the number that we have deleted. Then we look at the boxes immediately to the right and below and we slide the smaller number into the empty space, this is repeated until the empty spaces reaches the right hand border of the tableau. Then we fill the empty box with the negative of the number we emptied at the start. We repeat this process until we obtain a tableau filled with only negative numbers. Then we add the smallest number possible to each entry so that everything becomes positive again. Here is an example

$$\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array} \to \begin{array}{ccc} \bullet & 3 & 5 \\ 2 & 4 & \end{array} \to \begin{array}{ccc} 2 & 3 & 5 \\ \bullet & 4 & \end{array} \to \begin{array}{ccc} 2 & 3 & 5 \\ 4 & \bullet & \end{array} \to \begin{array}{ccc} 2 & 3 & 5 \\ 4 & -1 & \end{array}$$

$$\to \begin{array}{ccc} \bullet & 3 & 5 \\ 4 & -1 & \end{array} \to \begin{array}{ccc} 3 & \bullet & 5 \\ 4 & -1 & \end{array} \to \begin{array}{ccc} 3 & 5 & \bullet \\ 4 & -1 & \end{array} \to \begin{array}{ccc} 3 & 5 & -2 \\ 4 & -1 & \end{array}$$

$$\to \begin{array}{ccc} \bullet & 5 & -2 \\ 4 & -1 & \end{array} \to \begin{array}{ccc} 4 & 5 & -2 \\ \bullet & -1 & \end{array} \to \begin{array}{ccc} 4 & 5 & -2 \\ -3 & -1 & \end{array} \to \begin{array}{ccc} \bullet & 5 & -2 \\ -3 & -1 & \end{array}$$

$$\to \begin{array}{ccc} 5 & \bullet & -2 \\ -3 & -1 & \end{array} \to \begin{array}{ccc} 5 & -4 & -2 \\ -3 & -1 & \end{array} \to \begin{array}{ccc} \bullet & -4 & -2 \\ -3 & -1 & \end{array} \to \begin{array}{ccc} -5 & -4 & -2 \\ -3 & -1 & \end{array},$$

and finally we add 6 to make all the numbers positive and we obtain

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}.$$

The reader is encouraged to try another example on their own for maximum satisfaction! This transformation is described by another group, the *cactus group* which is linked to a geometric space called the *moduli space of stable curves*. The reader may like to turn to Figure 2.1 for a picture of a stable curve and a hint at why this group is given the name cactus group.

The aim of this thesis is to give a geometric interpretation of this combinatorial operation. The source of the geometry which realises this phenomenon comes from physics. Most readers would no doubt be aware of the great impact that mathematics has on physics, though unless you have been following more recent developments in mathematics you might not be aware that this tide has started to turn and in recent decades a tremendous amount of groundbreaking mathematics is being inspired by ideas from physics, most notably ideas from quantum field theory giving hints at deep phenomena in geometry. In the present case the geometric space we are interested in has its origins in the *Gaudin spin chain model*. This is a physical system that models the interactions of a number of charged particles on a line. A powerful technique called the *Bethe ansatz* allows one to describe the details of the geometric space produced by this system.

The motivation behind the problem of describing this phenomenon geometrically comes from a question posed by two mathematicians, C. Bonnafé and R. Rouquier. They attempt to give a new description of certain combinatorial objects called *cells* and in particular they conjecture that one should be able to describe these cells geometrically. For the general linear group these cells are related to tableaux and the cactus group. Though the work in this thesis does not answer the question of Bonnafé and Rouquier, it goes some way to demonstrating a connection between the two worlds.

Acknowledgements

My thanks go above all to my supervisors and teachers, Iain Gordon and Michael Wemyss. Both have been enormously generous with their time, support, ideas and mathematics during my time in Edinburgh. Due to them I have learnt to stop worrying and love geometry. I am very grateful to Iain for the initial suggestion of the problem considered in this thesis and for his continued input of ideas. I would like to thank David Jordan for his continued support and encouragement and the many members of the Hodge institute who have provided encouragement, interesting discussions and an engaging environment for learning new mathematics. Special thanks also to Becca, Chris and the many other students who made my time at Edinburgh enjoyable. Lastly, many thanks to my family and friends for their love and support, particularly my parents and my partner Petra.

Contents

Chapter 1

Introduction

The Hamiltonians for the Gaudin model are n commuting operators depending on distinct complex parameters $z_1, z_2, \ldots, z_n$ acting on an n-fold tensor product of irreducible representations of the *general linear Lie algebra* $\mathfrak{gl}_r$. The problem of understanding the spectrum of these operators (their simultaneous eigenspaces and corresponding eigenvalues) has received significant attention [Gau76; Gau83; FFR94; FFT10]. In this thesis we aim to understand how the spectrum changes as the parameters change. It will turn out that the monodromy of this system can be identified with certain natural transformations of crystals. Below we state the problem more precisely, explain a theorem which motivates a connection to crystal bases and outline our main strategy.

Let $\mathfrak{gl}_r$ be the Lie algebra of $r \times r$ matrices and e_{ij} the matrix with a 1 in the (i,j)-entry and 0 everywhere else. For $X \in U(\mathfrak{gl}_r)$, we let

$$X^{(a)} = 1 \otimes \cdots \otimes 1 \otimes X \otimes 1 \otimes \cdots \otimes 1 \in U(\mathfrak{gl}_r)^{\otimes n},$$

where X is placed in the a^{th} factor. Let $L(\lambda)$ be the irreducible $\mathfrak{gl}_r$-module with highest weight given by the partition λ. For $z = (z_1, z_2, \ldots, z_n)$ a set of distinct complex parameters, the *Gaudin Hamiltonians* are

$$H_a(z) = \sum_{b \neq a} \frac{\Omega_{ab}}{z_a - z_b} \in U(\mathfrak{gl}_r)^{\otimes n} \text{ where } \Omega_{ab} = \sum_{i,j} e_{ij}^{(a)} e_{ji}^{(b)}, \tag{1.0.1}$$

for $a = 1, 2, \ldots, n$. These operators act on the tensor product of representations $L(\lambda_\bullet) = L(\lambda_1) \otimes L(\lambda_2) \otimes \cdots \otimes L(\lambda_n)$ for an n-tuple of partitions $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with at most r rows.

These operators commute with the action of $\mathfrak{gl}_r$ on $L(\lambda_\bullet)$ (see [MTV10]). In particular this implies the Hamiltonians preserve weight spaces and singular (highest weight) vectors, thus the action of the operators $H_a(z)$ preserves the subspace $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$ for any partition μ. In fact, since $L(\lambda_\bullet)^{\mathsf{sing}}$ generates $L(\lambda_\bullet)$ as a $\mathfrak{gl}_r$-module, to understand the full action, it suffices to understand the restriction of the $H_a(z)$ to $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$.

1.1 Moduli of curves and the cactus group

Let $G(r;z) \subset U(\mathfrak{gl}_r)^{\otimes n}$ be the commutative subalgebra generated by the Gaudin Hamiltonians (1.0.1) (we will reserve the notation $G(z)$ for a slightly different but related object) for a set of parameters

$$z \in X_n = \{z \in \mathbb{C}^n \mid z_a \neq z_b \text{ for } a \neq b\}.$$

The affine group $\mathrm{Aff}_1 \simeq \mathbb{C}^\times \ltimes \mathbb{C}$ acts freely (provided $n \geq 2$) on X_n by simultaneous scaling and translation on each coordinate, i.e for two scalars $\alpha \in \mathbb{C}^\times$ and $\beta \in \mathbb{C}$

$$(\alpha, \beta) \cdot z = (\alpha z_1 + \beta, \alpha z_2 + \beta, \ldots, \alpha z_n + \beta).$$

The operator $H_a(\alpha z + \beta)$ is a scalar multiple of $H_a(z)$ and so the algebras $G(r; \alpha z + \beta)$ and $G(r;z)$ coincide. The parameter space we consider is thus X_n/Aff_1 which can be identified with $M_{0,n+1}(\mathbb{C})$, the moduli space of irreducible curves with $n+1$ marked points, up to isomorphism. The space $M_{0,n+1}(\mathbb{C})$ has a compactification $\overline{M}_{0,n+1}(\mathbb{C})$ which is the moduli space of *stable curves* with $n+1$ marked points.

Aguirre, Felder and Veselov showed in [AFV11] that the family of algebras $G^\circ(r)$ formed by $G(r;z)$ over $M_{0,n+1}(\mathbb{C})$ extends naturally to a family $G(r)$ of subalgebras of $U(\mathfrak{gl}_r)^{\otimes n}$ on $\overline{M}_{0,n+1}(\mathbb{C})$ (see Theorem 2.4.5 for a precise recollection of the result). The spectrum of these families of algebras acting on $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$ form finite maps over $M_{0,n+1}(\mathbb{C})$ and $\overline{M}_{0,n+1}(\mathbb{C})$ respectively; topologically, these are ramified covering spaces. To understand how the spectrum changes as z varies means to understand the monodromy of this covering space. By a theorem of Harris (see Appendix B), since these finite maps are birational to one another, they have the same monodromy. Concretely, we thus aim to give a description of the action of the monodromy group on the set of joint eigenspaces of $G(r;z)$ acting on $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$.

The fundamental group of the real points $\overline{M}_{0,n+1}(\mathbb{R})$ is the *pure cactus group* PJ_n (see [HK06] for a proof). As shown by Henriques and Kamnitzer [HK06] this group turns up in another context, namely as part of the monoidal structure on the category of $\mathfrak{gl}_r$-crystals. The category of $\mathfrak{gl}_r$-crystals has a tensor product and a commutor: a natural isomorphism $A \otimes B \cong B \otimes A$ for any two crystals A and B, however the category is not braided monoidal, instead the commutor obeys the *cactus relation*. This means the category is a *coboundary monoidal category*. Similar to braided monoidal categories, coboundary categories have a group acting on n-fold tensor products of objects, in the case of coboundary categories this group is the pure cactus group.

Let $\mathsf{B}(\lambda)$ be the irreducible crystal with highest weight given by the partition λ and let

$$\mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}} = [\mathsf{B}(\lambda_1) \otimes \mathsf{B}(\lambda_2) \otimes \cdots \otimes \mathsf{B}(\lambda_n)]_\mu^{\mathsf{sing}}.$$

Then PJ_n acts on $\mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$. This action can be described combinatorially. It is then natural to conjecture a link between the action of PJ_n on $\mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$ and the action of PJ_n by monodromy on the eigenspaces of $G(r;n)$ in $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$. Unfortunately this is not true, but only for a rather technical reason which we explain below. The precise conjecture which we prove in this thesis was stated in [Ryb14] and attributed to Etingof, however it was also made independently by Brochier-Gordon.

1.2 Main results

There is another reason why the real points $\overline{M}_{0,n+1}(\mathbb{R})$ enter our story. In the paper [MTV09a], Mukhin, Tarasov and Varchenko prove an intriguing result: the Gaudin Hamiltonians are diagonalisable on $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$ when $z \in M_{0,n+1}(\mathbb{R})$. For complex values of the parameters this is not true and it is a difficult problem to determine the locus where the Gaudin Hamiltonians are diagonalisable. The real points $M_{0,n+1}(\mathbb{R})$ is a disjoint union of simply connected pieces and thus provides no monodromy. One might hope to show the diagonalisability extends to the algebras over the whole of $\overline{M}_{0,n+1}(\mathbb{R})$ and indeed such a result is proven by Rybnikov in [Ryb14]. If the number of $G(r;z)$ simultaneous eigenspaces is constant for $z \in \overline{M}_{0,n+1}(\mathbb{R})$ we would have a topological covering space and we can hope to calculate the monodromy. Unfortunately this is not true. At certain points, the number of simultaneous eigenspaces can drop, as we demonstrate in Section 2.4.4. Another way to say this is that the image of $G(r;z)$ in $\mathrm{End}(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$ drops dimension. Since it is still diagonalisable there must be a maximal commutative subalgebra containing $G(r;z)$.

In [FFR94] a maximal commutative subalgebra $A(\lambda_\bullet;z)_\mu \subset \mathrm{End}(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$ containing the Gaudin Hamiltonians is constructed, called the *Bethe algebra* for $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$. As is shown in Lemma 3.1.7, these satisfy $A(\lambda_\bullet;z)_\mu = A(\lambda_\bullet;\alpha z+\beta)_\mu$. This produces a family of algebras $A(\lambda_\bullet)_\mu$ over $M_{0,n+1}(\mathbb{C})$. We denote the spectrum of this family by

$$\pi_{\lambda_\bullet,\mu} : \mathcal{A}(\lambda_\bullet)_\mu = \operatorname{Spec} A(\lambda_\bullet)_\mu \longrightarrow M_{0,n+1}(\mathbb{C}).$$

The morphism $\pi_{\lambda_\bullet,\mu}$ is finite (i.e. it is a finite ramified covering space) and is our main object of study. Let $\mathrm{Gal}(\pi_{\lambda_\bullet,\mu})$ be the *Galois group* of $\pi_{\lambda_\bullet,\mu}$, this is the algebraic version of the monodromy group (see Appendix B) and its action on a generic fibre of $\mathrm{Gal}(\pi_{\lambda_\bullet,\mu})$ coincides with the action of the fundamental group of any open, dense subset of $M_{0,n+1}(\mathbb{C})$ by monodromy. Denote the fibre of $\pi_{\lambda_\bullet,\mu}$ over $z \in M_{0,n+1}(\mathbb{C})$ by $\mathcal{A}(\lambda_\bullet;z)_\mu$. The following is a precise statement of the first main theorem of this thesis.

Theorem A. *There exists a homomorphism $PJ_n \to \mathrm{Gal}(\pi_{\lambda_\bullet,\mu})$ from the pure cactus group to the Galois group of $\pi_{\lambda_\bullet,\mu}$ and a bijection*

$$\mathcal{A}(\lambda_\bullet, z)_\mu \longrightarrow \mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}},$$

equivariant for the induced action of PJ_n.

In the case $r = 2$, this theorem was proved by Rybnikov in [Ryb14]. It also follows from calculations made by Varchenko in [Var95]. This theorem appears in [Whi15].

Upon restriction to the case when $\lambda_\bullet = (\Box^n)$, where $\Box$ is the partition (1), both sets in Theorem A are naturally labelled by standard tableaux of shape μ. The representation $V = L(\Box)$ is the *vector representation* for $\mathfrak{gl}_r$ and is isomorphic to $\mathbb{C}^r$. By Schur-Weyl duality, the space $[V^{\otimes n}]_\mu^{\mathsf{sing}}$ is isomorphic to the irreducible representation for the symmetric group S_n which acts by permutation of the tensor factors. Let $z = (z_1, z_2, \ldots, z_n)$ be an n-tuple of distinct real numbers such that $z_1 < z_2 \ldots < z_n$, we take a limit as $z_i \to \infty$ for all i such that $z_i/z_{i+i} \to 0$. The limit of the Gaudin Hamiltonians coincides with the *Jucys-Murphy elements* whose simultaneous eigenspaces are one dimensional and canonically labelled by standard μ-tableaux. The spectrum of the Jucys-Murphy operators was first calculated by Jucys in [Juc74] and subsequently rediscovered by Murphy in [Mur83]. The important role of these operators in the representation theory of the symmetric group was developed by Okounkov and Vershik in the paper [OV96].

More precisely, $\lim_{z\to\infty} z_a H_z(z) = L_a$ where L_a is the Jucys-Murphy element given by the sum of transpositions $\sum_{b<a}(a,b)$. The eigenvalues of the Jucys-Murphy elements are described by standard tableaux. If S is a standard tableaux, then let $c_S(a)$ be the row number minus the column number of the box containing a. Then if v is an simultaneous eigenvector for the L_a then $L_a v = c_S(a)v$ for a unique standard tableaux S, see Theorem 3.2.22. Given a closed point $\chi \in \mathcal{A}(z)_\mu = \mathcal{A}(\Box^n; z)_\mu$, we will consider it as a ring homomorphism $\chi : A(z)_\mu \longrightarrow \mathbb{C}$.

On the other hand, the crystal $\mathtt{B} = \mathtt{B}(\Box)$ is the set $\{1, 2, \ldots, r\}$ so $\mathtt{B}^{\otimes n}$ can be identified with the set of words of length n in the letters $\{1, 2, \ldots, r\}$. The *RSK correspondence* allows us to label the elements of $[\mathtt{B}^{\otimes n}]_\mu^{\mathsf{sing}}$ by standard μ-tableaux. The bijection from Theorem A gives a bijection

$$\mathbb{X}_\mu(z) : \mathcal{A}(z)_\mu \longrightarrow [\mathtt{B}^{\otimes n}]_\mu^{\mathsf{sing}} \xrightarrow{\mathtt{RSK}} \mathtt{SYT}(\mu).$$

Theorem B. *For $z = (z_1, z_2, \ldots, z_n)$ an n-tuple of distinct real numbers such that $z_1 < z_2 < \ldots < z_n$, the bijection $\mathbb{X}_\mu(z) : \mathcal{A}(z)_\mu \longrightarrow \mathtt{SYT}(\mu)$ is given by $\mathbb{X}_\mu(z)(\chi) = S$, where S is the unique tableau with*

$$c_S(a) = \lim_{z\to\infty} \chi_S(z_a H_a(z)).$$

1.3 Calogero-Moser space

The main motivation for the problem considered in this thesis is the special case when $\lambda_\bullet = (\Box^n)$, $r = n$ and $\mu = (1, 1, \ldots, 1)$. Consider a more general version of the Gaudin Hamiltonians, depending on an additional set of complex parameters $q = (q_1, q_2, \ldots, q_n) \in \mathbb{C}^n$,

$$H_a(z) = \sum_{i=1}^{n} q_i e_{ii}^{(a)} + \sum_{b \neq a} \frac{\Omega_{ab}}{z_a - z_b} \in U(\mathfrak{gl}_r)^{\otimes n}, \tag{1.3.1}$$

These *generalised Gaudin Hamiltonians* act on $V^{\otimes n}$ however do not commute with the action of $\mathfrak{gl}_n$ so we cannot restrict the action to the space of singular vectors. However, the operators do commute with the action of $\mathfrak{h} \subset \mathfrak{gl}_n$, the Cartan subalgebra of diagonal matrices, thus we can restrict their action to $W = [V^{\otimes n}]_{(1,1,\ldots,1)}$. Note that $\dim W = n!$.

The symmetric group S_n acts on $\mathfrak{h}$ by reflections and hence on $\mathfrak{h}^* \oplus \mathfrak{h}$ by symplectic reflections. The *rational Cherednik algebra* $\mathcal{H}_{t,c}(S_n)$ is an algebra depending on parameters $t, c \in \mathbb{C}$. When $t = 0$ this algebra deforms the skew group algebra $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[\mathfrak{h}] \rtimes S_n$ and has a large centre Z_c. The centre contains $\mathbb{C}[\mathfrak{h}^*]^{S_n} \otimes \mathbb{C}[\mathfrak{h}]^{S_n}$ and is a free module of rank $n!$ over this subalgebra. Etingof and Ginzburg in [EG02] proved the spectrum of Z_c is isomorphic to the *Calogero-Moser space*,

$$CM_n = \{(X, Y) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \,|\, \mathrm{rank}([X, Y] + I_n) = 1\} / GL_n,$$

where GL_n acts by simultaneous conjugation. The Calogero-Moser space has a natural map π to $\mathbb{C}^{(n)} \times \mathbb{C}^{(n)}$ (where $\mathbb{C}^{(n)} = \mathbb{C}^n / S_n$) given by sending (X, Y) to (z, q) where z is the (unordered) collection of eigenvalues of X and q the eigenvalues of Y. As a consequence of Etingof and Ginzburg's result, the map is finite of degree $n!$. In [MTV14], Mukhin, Tarasov and Varchenko show the Calogero-Moser space is given by the spectrum of the operators (1.3.1) acting on W. In particular, if we consider the fibre of π over $X_n/S_n \times \{0\}$ this is given by the spectrum of the ordinary Hamiltonians (1.0.1), up to a quotient by the symmetric group. The original observation that a relation exists between the Knizhnik-Zamolodchikov equations and the Calogero-Moser integrable system goes back to Matsuo [Mat92] and Cherednik [Che94].

More generally there exists a rational Cherednik algebra for a Coxeter group of any type and we can consider its centre as a generalisation of Calogero-Moser space. Bonnafé and Rouquier in [BR13] conjecture and provide evidence for a close link between the geometry of Calogero-Moser space and the Kazhdan-Lusztig cells of the associated Coxeter group in all types. In particular it is conjectured the Kazhdan-Lusztig cells are produced as the orbits of a Galois group action. Theorems A and B provide evidence that the Kazhdan-Lusztig cells can in fact be recovered from the Galois theory of Calogero-Moser space in type A. This is justified by the following observation. As far as the author is aware, this was first noticed by Gordon.

As mentioned above, we can identify the set $\mathtt{B}^{\otimes n}$ with words of length n in the letters $\{1, 2, \ldots, n\}$ and thus $[\mathtt{B}^{\otimes n}]_{(1,1\ldots,1)}$ with the symmetric group S_n. A small calculation shows that the PJ_n-orbits on $[\mathsf{B}^{\otimes n}]^{\mathsf{sing}}_{(1,1\ldots,1)}$ are exactly the Kazhdan-Lusztig cells. Theorem A says this action (and therefore the orbits of the action) are the same as the action of the Galois group of the spectrum of the Gaudin Hamiltonians which we have just seen is related to Calogero-Moser space.

1.4 Outline

We briefly summarise here the content of this thesis. Chapters 2, 3 and 4 contain mostly background material. A small number of results and examples are original and these are listed below. All results and proofs in Chapter 5 (with the exception of Section 5.5.2) and Appendix A are original. Appendix C contain results which the author failed to find in the literature, but are almost certainly known, and for the sake of completeness we have included detailed proofs.

Chapter 2 recalls the definitions of the Gaudin Hamiltonians, Gaudin subalgebras and the moduli spaces of stable curves. The observations made in Section 2.2.7 are original as is the statement of Conjecture 2.4.11. In Chapter 3 we recall the definition of Bethe algebras and crystals. We use the results stated in this chapter to formulate the two main theorems which we prove in Chapter 5.

Chapter 4 recalls some basic facts from Schubert calculus and Speyer's compact family of Schubert intersections. This is the main technical tool of the thesis and we recall the combinatorial description of its real points. We also recall the relationship between the Bethe algebras and Schubert calculus in Corollary 4.7.5; whilst this statement has not appeared in the literature, it follows directly from work of Mukhin, Tarasov and Varchenko and some standard arguments in algebraic geometry. Next we recall the formulation of the algebraic Bethe ansatz and the labelling of certain critical points by standard tableaux.

In Chapter 5 we prove the main results of the thesis. The first step towards this goal is to calculate the equivariant monodromy of Speyer's family. We prove by combinatorial means, that the equivariant monodromy is given by the action of the cactus group on crystals. We are then able to identify this with monodromy in the Bethe spectrum using results from Chapter 4. This proves Theorem A. To prove Theorem B we first reinterpret the labelling of the fibres of Speyer's family, we then use the connection to critical points to show this labelling matches our labelling of the Bethe spectrum. The contents of this chapter are original with the exception of Section 5.5.2.

Chapter 2

Gaudin Hamiltonians, stable curves and Gaudin subalgebras

This chapter provides background about the Gaudin Hamiltonians and moduli of stable curves. We recall a result of Aguirre, Felder and Veselov [AFV11] which says algebras generated by the Gaudin Hamiltonians fit into a larger family of algebras over the moduli space of stable curves with $n+1$ marked points. While we will not use this result in any integral way in this thesis, it provides motivation for the link between the Gaudin Hamiltonians and the cactus group which will also be introduced in the next chapter. This result will also provide an important example in Section 2.2.7 which motivates the need to consider the more general Bethe algebras.

2.1 Notation and preliminaries

This thesis will use the following conventions.

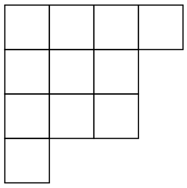
- The *symmetric group* on n letters is denoted S_n. It is the group of permutations of the set $[n] = \{1, 2, \ldots, n\}$. Permutations are thought of as functions and thus are multiplied from right to left, so if $\sigma = (1, 2)$ and $\tau = (2, 3)$, written in cycle notation, then $\sigma\tau = (1, 2, 3)$.

- The *general linear Lie algebra* is denoted $\mathfrak{gl}_r$. It is the set of $r \times r$ matrices with Lie bracket given by the commutator.

- The *universal enveloping algebra* for the Lie algebra $\mathfrak{gl}_r$ is denoted $U(\mathfrak{gl}_r)$.

- The complement of the type A hyperplane arrangement in $\mathbb{C}^n$ is denoted X_n. Explicitly
$$X_n = \{ z \in \mathbb{C}^n \,|\, z_i \neq z_j \text{ for any } i \neq j \},$$
is the set of n-tuples of distinct complex numbers.

- The set of *partitions* is denoted `Part`. Recall that a partition of a natural number n is a sequence of non-increasing natural numbers, whose sum is n. For example $(4,3,3,1)$ is a partition of 11. A partition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ is represented by a diagram consisting of left aligned rows of boxes, the i^{th} row having $\lambda^{(i)}$ boxes. This will be called the *shape* or *diagram* of λ. For example $(4,3,3,1)$ has

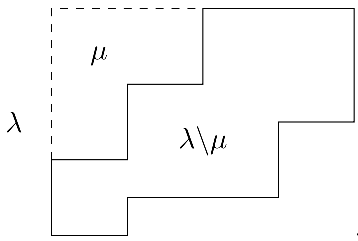


  as its diagram.

- The partition (1) will usually be denoted $\square$
- The set of partitions of n is denoted $\texttt{Part}_n$.
- The set of partitions with at most r rows is denoted $\texttt{Part}(r)$.
- The set of partitions with at most r rows and $d-r$ columns is denoted $\texttt{Part}(r,d)$.

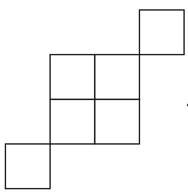
Remark 2.1.1. Our indexing of the parts of a partition using upper indices is non-standard, however very often we will be dealing with *sequences* of partitions, denoted $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, and our notation will be less cluttered if we allow ourselves to use lower indices for the sequence.

Given two partitions λ and μ, we say that λ *contains* μ, denoted $\mu \subseteq \lambda$ if $\mu^{(i)} \leq \lambda^{(i)}$ for all $i \geq 1$. Diagrammatically this corresponds to the situation when the diagram for μ can be placed completely within the diagram for λ. When $\mu \subseteq \lambda$, we can form the *skew-shape* $\lambda \backslash \mu$ which is the shape λ with the boxes corresponding to μ removed. Pictorially $\lambda \backslash \mu$ is



.

The skew-shape $(4,3,3,1)\backslash(3,1,1)$ is



.

We denote the number of boxes in $\lambda\backslash\mu$ by $|\lambda\backslash\mu|$. We will say a skew-shape $\lambda\backslash\mu$ is *normal* if $\mu = \emptyset$. That is, if it is the north-western part of a rectangle.

2.2 The Kohno-Drinfeld algebra and Gaudin Hamiltonians

Our aim in this chapter is to discuss the result of Aguirre, Felder and Veselov which shows the algebras generated by the Gaudin Hamiltonians are a subset of a larger moduli space of similar commutative algebras. To state this precisely we follow [AFV11] and consider the Gaudin Hamiltonians as elements of the *Kohno-Drinfeld Lie algebra*. Whilst this is not strictly necessary, it makes the statements less cumbersome. This section recalls the Kohno-Drinfeld Lie algebra and states precisely the finite maps whose Galois theory we aim to understand. At the end of the section we provide examples which show in general these maps can have ramification and the Gaudin Hamiltonians need not be diagonalisable.

2.2.1 The Kohno-Drinfeld Lie algebra

Definition 2.2.1 ([Koh83]). The *Kohno-Drinfeld Lie algebra* $\mathfrak{t}_n$ is the Lie algebra generated by symbols $t_{ij} = t_{ji}$ for $1 \leq i \neq j \leq n$ with relations

$$[t_{ij}, t_{kl}] = 0 \quad \text{if } i, j, k, l \text{ are distinct,} \tag{2.2.1}$$

$$[t_{ij}, t_{ik} + t_{jk}] = 0 \quad \text{if } i, j, k \text{ are distinct.} \tag{2.2.2}$$

Let the generators t_{ij} have degree 1, since the relations are homogeneous, the Kohno-Drinfeld Lie algebra is graded.

Definition 2.2.2. For each $z \in X_n$ we define the *Gaudin Hamiltonians* at z to be the following elements of $\mathfrak{t}_n$:

$$H_a(z) = \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{t_{ab}}{z_a - z_b} \quad \text{for } a = 1, 2, \ldots, n,$$

and $G(z)$ the Lie subalgebra generated by $H_a(z)$ for $a = 1, 2, \ldots, n$.

Lemma 2.2.3. *For any $z \in X_n$, the subspace $G(z)$ is a $(n-1)$-dimensional, abelian, Lie subalgebra, homogeneous of degree* 1.

Proof. We can directly compute the commutator of the generators. If $a \neq b$,

$$[H_a(z), H_b(z)] = \left[\sum_{c\neq a} \frac{t_{ac}}{z_a - z_c}, \sum_{d\neq b} \frac{t_{bd}}{z_b - z_d} \right]$$

$$
\begin{aligned}
&= \sum_{c \neq a} \sum_{d \neq b} \frac{1}{(z_a - z_c)(z_b - z_d)} [t_{ac}, t_{bd}] \\
&= \sum_{\substack{c,d \notin \{a,b\} \\ c \neq d}} \frac{1}{(z_a - z_c)(z_b - z_d)} [t_{ac}, t_{bd}] \\
&\quad + \sum_{c = d \notin \{a,b\}} \frac{1}{(z_a - z_c)(z_b - z_c)} [t_{ac}, t_{bc}] \\
&\quad + \sum_{\substack{c \notin \{a,b\} \\ d = a}} \frac{1}{(z_a - z_c)(z_b - z_a)} [t_{ac}, t_{ba}] \\
&\quad + \sum_{\substack{d \notin \{a,b\} \\ c = b}} \frac{1}{(z_a - z_b)(z_b - z_d)} [t_{ab}, t_{bd}] \\
&\quad - \frac{1}{(z_a - z_b)^2} [t_{ab}, t_{ba}]
\end{aligned}
$$

the first group of terms of the right hand side is zero by (2.2.1), we can transform the second group of terms using $[t_{ac}, t_{bc}] = -[t_{ac}, t_{ab}]$ by (2.2.2), and the last term vanishes by skew symmetry, so

$$
\begin{aligned}
&= \sum_{c \notin \{a,b\}} \frac{1}{z_a - z_c} \left(-\frac{1}{z_b - z_c} + \frac{1}{z_b - z_a} \right) [t_{ac}, t_{ab}] \\
&\quad + \sum_{d \notin \{a,b\}} \frac{1}{(z_a - z_b)(z_b - z_d)} [t_{ab}, t_{bd}] \\
&= \sum_{c \notin \{a,b\}} \frac{-1}{(z_b - z_c)(z_a - z_b)} [t_{ac}, t_{ab}] \\
&\quad + \sum_{d \notin \{a,b\}} \frac{1}{(z_a - z_b)(z_b - z_d)} [t_{ab}, t_{bd}] \\
&= \sum_{c \notin \{a,b\}} \frac{1}{(z_b - z_c)(z_a - z_b)} [t_{ab}, t_{ac} + t_{bc}]
\end{aligned}
$$

which is zero again by (2.2.1). This shows $G(z)$ is an abelian Lie subalgebra. Since it is generated by commuting homogeneous elements of degree 1 the algebra is homogeneous of degree 1. The fact $G(z)$ is abelian means a subset of $\{H_a(z)\}_a$ will form a basis. Consider $K = \sum_{a=1}^n w_a H_a(z)$ for $w \in \mathbb{C}^n$. Thus

$$
K = \sum_{a=1}^{n} \sum_{b \neq a} \frac{w_a}{z_a - z_b} t_{ab} \tag{2.2.3}
$$

$$
= \sum_{1 \leq a < b \leq n} \frac{w_a - w_b}{z_a - z_b} t_{ab}. \tag{2.2.4}
$$

So $K = 0$ if and only if $w_a = w_b$ for all pairs $1 \leq a, b \leq n$. Thus, the Lie algebra $G(z)$ is $(n-1)$-dimensional. $\square$

2.2.2 A homomorphism to $U(\mathfrak{gl}_r)^{\otimes n}$

Let $e_{ij} \in \mathfrak{gl}_r$ denote the $(r \times r)$-matrix with a 1 in the (i,j)-position and zeros elsewhere, and consider the Casimir-type operator $\Omega \in U(\mathfrak{gl}_r) \otimes U(\mathfrak{gl}_r)$ defined by

$$\Omega = \sum_{1 \leq i,j \leq n} e_{ij} \otimes e_{ji}.$$

For an algebra A, we will use the notation $x \mapsto x^{(a)}$ to denote the embedding $A \to A^{\otimes n}$ where $x \in A$ is sent to $\mathrm{id}^{\otimes a-1} \otimes x \otimes \mathrm{id}^{\otimes n-a}$. Similarly for any $c = \sum x_i \otimes y_i \in A \otimes A$ we use $c \mapsto c^{(ab)}$ to denote the embedding sending c to $\sum \mathrm{id}^{\otimes a-1} \otimes x_i \otimes \mathrm{id}^{\otimes b-a-1} \otimes y_i \otimes \mathrm{id}^{n-b}$. For every $r > 0$ there is an algebra homomorphism

$$\varphi_r : U(\mathfrak{t}_n) \longrightarrow U(\mathfrak{gl}_r)^{\otimes n},$$

given by sending t_{ab} to $\Omega^{(ab)}$. The algebra $G(r; z)$ considered in the introduction is the image of $G(z)$ under φ_r. Any representation of $U(\mathfrak{gl}_r)^{\otimes n}$ pulls back along φ_r to a representation of $U(\mathfrak{t}_n)$.

2.2.3 The affine group and moduli of curves

Definition 2.2.4. An *irreducible, genus zero curve with k marked points*, (C, z), is a complex curve, C, isomorphic to $\mathbb{P}^1$ together with a collection $z = (z_1, z_2, \ldots, z_k)$ of distinct points of C. We say two such curves, (C, z) and (D, w), are *equivalent* if there exists an isomorphism $\varphi : C \longrightarrow D$ identifying marked points, i.e. $\varphi(z_i) = w_i$ for $i \in [k]$.

In order to understand the moduli space of such curves we use the following fundamental fact about automorphisms of the projective line.

Proposition 2.2.5. *The automorphism group of $\mathbb{P}^1$ is given by PGL_2, where the action is given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [x : y] = [ax + by : cx + dy]$$

for some choice of homogeneous coordinates on $\mathbb{P}^1$. Given any triple of distinct points $z_1, z_2, z_3 \in \mathbb{P}^1$ there is a unique automorphism such that

$$z_1 \mapsto 0, z_2 \mapsto 1, \text{ and } z_3 \mapsto \infty.$$

Proof. The first claim is demonstrated in (for example) [GH94, Section 0.4]. Once it is established that $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2$ the latter claim is a simple calculation. Identify $z \in \mathbb{C} \subset \mathbb{P}^1$ with $[1 : z]$ in our homogeneous coordinates, and let $\phi \in \mathrm{Aut}(\mathbb{P}^1)$ be given by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2 .$$

Assume for simplicity $z_1, z_2, z_3 \in \mathbb{C} \subset \mathbb{P}^1$. The condition $\phi(z_1) = 0, \phi(z_2) = 1$ and $\phi(z_3) = \infty$ mean

$$\begin{aligned}
[a + bz_1 : c + dz_1] &= [1 : 0] \\
[a + bz_2 : c + dz_2] &= [1 : 1] \\
[a + bz_3 : c + dz_3] &= [0 : 1].
\end{aligned}$$

So we have the relations $c = -dz_1$, $a = -bz_3$ and $(a - c) = (d - b)z_2$. Putting these together, the matrix is

$$\begin{pmatrix} \frac{z_2 - z_1}{z_3 - z_2} z_3 d & \frac{z_1 - z_2}{z_3 - z_2} d \\ -dz_1 & d \end{pmatrix}.$$

Since the determinant is $(z_2 - z_1)(z_3 - z_1)d^2/(z_3 - z_2)$ and the points z_1, z_2, z_3 are distinct, this is a unique element of PGL_2. The case when one of the points is ∞ follows similarly. $\square$

Let $\Delta \subset (\mathbb{P}^1)^k$ be the *big diagonal*, that is Δ is the union of all hyperplanes $H_{ij} \subset (\mathbb{P}^1)^k$, where $H_{ij} = \left\{ z \in (\mathbb{P}^1)^k \,\middle|\, z_i = z_j \right\}$.

Proposition 2.2.6. *When $k \geq 3$, there exists a fine moduli space of irreducible, genus zero curves with k marked points, denoted $M_{0,k}(\mathbb{C})$. It is isomorphic to the quotient $((\mathbb{P}^1)^k - \Delta)/\,\mathrm{PGL}_2$ where PGL_2 acts diagonally on $(\mathbb{P}^1)^k$.*

Proof. For a rigorous treatment of this moduli problem see [KV07, Chapter 0]. Informally, we can choose for any irreducible, genus zero curve C with k marked points, an isomorphism $\theta : C \longrightarrow \mathbb{P}^1$. If $z_1, z_2, \ldots, z_k \in C$ are the marked points we identify C with the point $(\theta(z_1), \ldots, \theta(z_k)) \in (\mathbb{P}^1)^k - \Delta$. This point is only well defined up to the action of PGL_2. By definition, curves represented by distinct points in $(\mathbb{P}^1)^k - \Delta$, related by the action of PGL_2, are isomorphic as irreducible, genus zero curves with k marked points. $\square$

The *affine group* Aff_1 is the group of automorphisms of $\mathbb{C}^n$ generated by simultaneous translations

$$(z_1, z_2, \ldots, z_n) \mapsto (z_1 + \beta, z_2 + \beta, \ldots, z_n + \beta),$$

for a complex number $\beta \in \mathbb{C}$, and simultaneous scaling

$$(z_1, z_2, \ldots, z_n) \mapsto (\alpha z_1, \alpha z_2, \ldots, \alpha z_n),$$

for a nonzero complex number $\alpha \in \mathbb{C}^\times$. The space $X_n \subset \mathbb{C}^n$ is preserved by the action of Aff_1. Furthermore if $n \geq 2$, the action of Aff_1 is free on X_n, thus it makes sense to consider the quotient space.

Proposition 2.2.7. *The algebras $G(z)$ are unchanged by the action of* Aff_1, *that is* $G(\alpha z+\beta)=G(z)$.

Proof. This is simple to see since $H_a(\alpha z+\beta)=\frac{1}{\alpha}H_a(z)$. □

Proposition 2.2.8. *The moduli space $M_{0,n+1}(\mathbb{C})$ of irreducible, genus zero curves with* $n+1$ marked points *is isomorphic to the quotient X_n/Aff_1.*

Proof. By Propostion 2.2.6 we need to identify the quotient X_n/Aff_1 with the quotient $((\mathbb{P}^1)^{n+1}-\Delta)/\operatorname{PGL}_2$. Construct a morphism

$$X_n \longrightarrow ((\mathbb{P}^1)^n-\Delta)/\operatorname{PGL}_2$$

by sending the point $(z_1,\dots,z_n)$ to the orbit of the point $(z_1,\dots,z_n,\infty)$. This is an Aff_1 equivariant morphism, since the points $(z_1,\dots,z_n,\infty)$ and $(\alpha z_1+\beta,\dots,\alpha z_n+\beta,\infty)$ are in the same PGL_2 orbit. Thus we have a well defined morphism which is invertible. □

This allows us to think of the algebras $G(z)$ as being parameterised by points $z\in M_{0,n+1}(\mathbb{C})$. We will often abuse notation and identify a point $z\in X_n$ with its equivalence class in $M_{0,n+1}(\mathbb{C})$.

2.2.4 The Gaudin spectrum

There is a vector bundle on $M_{0,n+1}(\mathbb{C})$, denoted G°, whose fibre over z is $G(z)$ (the existence is verified by Theorem 2.4.5). Let $\mathcal{G}^\circ$ be the (locally free, coherent) sheaf of sections on G°. Fix a representation W of $U(\mathfrak{gl}_r)^{\otimes n}$. The homomorphism φ_r induces a morphism of locally free, coherent sheaves

$$\mathcal{G}^\circ \longrightarrow \mathcal{E}nd(W),$$

where $\mathcal{E}nd(W)$ is the sheaf of sections for the trivial bundle with fibre $\operatorname{End}(W)$. By the universal property of the symmetric algebra there is a morphism

$$\operatorname{Sym}\mathcal{G}^\circ \longrightarrow \mathcal{E}nd(W)$$

whose image we denote $\mathcal{G}^\circ_W$. The fibre of this coherent (not necessarily locally free) sheaf over $z\in M_{0,n+1}(\mathbb{C})$ is the commutative subalgebra of $\operatorname{End}(W)$ generated by $G(z)$. Thus $\mathcal{G}^\circ_W$ is a sheaf of commutative algebras on $M_{0,n+1}(\mathbb{C})$, and it is natural to consider its spectrum

$$\pi^\circ_W : \operatorname{Spec}\mathcal{G}^\circ_W \longrightarrow M_{0,n+1}(\mathbb{C}).$$

Since $\mathcal{G}^\circ_W$ is a coherent sheaf of algebras, π°_W is a finite map.

Definition 2.2.9. The finite map π°_W is the *Gaudin spectrum* associated to the representation W.

The Gaudin spectrum encodes the simultaneous eigenvalues of the Gaudin Hamiltonians as z varies. The main question investigated in this thesis is to understand the Galois theory of the Gaudin spectrum for tensor products of irreducible $\mathfrak{gl}_r$-representations. A brief recollection of what we mean by the Galois theory of a finite map is given in Appendix B. Informally, since π_W° is finite, it is a ramified cover of $M_{0,n+1}(\mathbb{C})$. We would like to understand the monodromy of this map. However, since π_W° is not necessarily a genuine covering space (we show this is the case in Section 2.2.7), we need to exclude points of ramification in order to make this meaningful. The Galois theory of the associated field extension encodes this monodromy algebraically.

2.2.5 The $\mathfrak{gl}_r$-action and tensor product representations

In this section we will discuss the $\mathfrak{gl}_r$-representations on which we will study the action of the Gaudin Hamiltonians. If $\lambda \in \mathtt{Part}(r)$, let $L(\lambda)$ denote the finite dimensional, irreducible $\mathfrak{gl}_r$-representation corresponding to the partition λ. Given a sequence $\lambda_\bullet = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of partitions, we denote the tensor product of the corresponding representations by

$$L(\lambda_\bullet) = L(\lambda_1) \otimes L(\lambda_2) \otimes \cdots \otimes L(\lambda_n).$$

Proposition 2.2.10. *The image of $\mathfrak{t}_n$ (and in particular $G(z)$) under φ_r commutes with the image of $U(\mathfrak{gl}_r)$ in $U(\mathfrak{gl}_r)^{\otimes n}$ under the iterated coproduct. In particular, if $W = \bigotimes_{i=1}^n M_i$ is a tensor product of $\mathfrak{gl}_r$ representations, the action of $\mathfrak{t}_n$ and thus $G(z)$ commutes with the action of $\mathfrak{gl}_r$.*

Proof. We will show Ω commutes with the image of the basis elements e_{ij} in $U(\mathfrak{gl}_r)^{\otimes 2}$ under the coproduct. We have $\Delta(e_{ij}) = e_{ij} \otimes 1 + 1 \otimes e_{ij}$. Thus

$$\begin{aligned}
[e_{ij}, \Omega] = (e_{ij} \otimes 1 + 1 \otimes e_{ij}) &\left(\sum_{k,l} e_{kl} \otimes e_{lk} \right) \\
&- \left(\sum_{k,l} e_{kl} \otimes e_{lk} \right) (e_{ij} \otimes 1 + 1 \otimes e_{ij}) \\
&= \sum_{k,l} e_{ij} e_{kl} \otimes e_{lk} + \sum_{k,l} e_{kl} \otimes e_{ij} e_{lk} - \sum_{k,l} e_{kl} e_{ij} \otimes e_{lk} - \sum_{k,l} e_{kl} \otimes e_{lk} e_{ij} \\
&= \sum_{k,l} \delta_{jk} e_{il} \otimes e_{lk} + \sum_{k,l} \delta_{jl} e_{kl} \otimes e_{ik} - \sum_{k,l} \delta_{li} e_{kj} \otimes e_{lk} - \sum_{k,l} \delta_{ki} e_{kl} \otimes e_{lj} \\
&= \sum_{l} e_{il} \otimes e_{lj} + \sum_{k} e_{kj} \otimes e_{ik} - \sum_{k} e_{kj} \otimes e_{ik} - \sum_{l} e_{il} \otimes e_{lj} = 0.
\end{aligned}$$

In the general case, the same proof holds since we only ever act on two out of the n tensor factors. $\square$

Let $W = \bigotimes_{i=1}^{n} M_i$ be a tensor product of $\mathfrak{gl}_r$-representations and denote the *Cartan subalgebra* by $\mathfrak{h} = \bigoplus_{1\leq i\leq r} \mathbb{C}e_{ii}$ the . As a consequence of Proposition 2.2.10, the algebras $G(z)$ leave the weight spaces

$$W_\alpha = \{v \in V \,|\, hv = \alpha(h)v, \text{ for } h \in \mathfrak{h}\}$$

invariant, for any $\alpha \in \mathfrak{h}^*$. The algebra $G(z)$ also leaves invariant the space of *singular* (or *highest weight*) vectors

$$W^{\mathsf{sing}} = \{v \in V \,|\, e_{ij}v = 0 \text{ for any } i < j\}\,.$$

The vector space $W^{\mathsf{sing}} = \bigoplus_{\mu\in\mathfrak{h}^*} W_\mu^{\mathsf{sing}}$ generates W as a $\mathfrak{gl}_r$-module. Since $\Omega^{(ab)}$ commutes with the action of $\mathfrak{gl}_r$ on W, the spectrum of $\Omega^{(ab)}$ on W is determined by its spectrum on W^{sing}, thus we can restrict attention to the action of $G(z)$ on this subspace. Furthermore, the decomposition of W^{sing} into weight spaces is compatible with the decomposition into $G(z)$-eigenspaces. Thus we will restrict our attention to studying the action of $G(z)$ on the subspaces W_μ^{sing} of singular vectors of a fixed weight μ. The main objects of study will be the representations

$$L(\lambda_\bullet)_\mu^{\mathsf{sing}} = [L(\lambda_1) \otimes L(\lambda_2) \otimes \cdots \otimes L(\lambda_n)]_\mu^{\mathsf{sing}}.$$

Since μ is the weight of a singular vector, it must be a partition of $|\lambda_\bullet|$. We will denote the Gaudin spectrum corresponding to $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$ by $\pi^\circ_{\lambda_\bullet,\mu}$.

2.2.6 Schur-Weyl duality

Schur-Weyl duality relates representations of $\mathfrak{gl}_r$ with representations of the symmetric group S_n. Let $S(\mu)$ denote the irreducible $\mathbb{C}S_n$-module corresponding to $\mu \in \mathtt{Part}_n$. Let $V = L(\square) \cong \mathbb{C}^r$ denote the *vector representation* of $\mathfrak{gl}_r$, then S_n acts on $V^{\otimes n}$ by permuting the tensor factors, that is

$$\sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

If $r \geq n$ then the representation $\mathbb{C}S_n \to \mathrm{End}(V^{\otimes n})$ is faithful.

Theorem 2.2.11 (Schur-Weyl duality)**.** *As a $U(\mathfrak{gl}_r) \otimes \mathbb{C}S_n$ representation, $V^{\otimes n}$ has the decomposition,*

$$V^{\otimes n} \cong \bigoplus_{\lambda} L(\lambda) \otimes S(\lambda). \tag{2.2.5}$$

The sum is indexed over partitions $\lambda \in \mathtt{Part}(r)$ with $|\lambda| = n$. If

$$\rho : \mathbb{C}S_n \longrightarrow \mathrm{End}(V^{\otimes n})$$
$$\eta : U(\mathfrak{gl}_r) \longrightarrow \mathrm{End}(V^{\otimes n}),$$

are the respective representations then

$$\operatorname{im}\rho = \operatorname{End}_{\mathfrak{gl}_r}(V^{\otimes n}) \text{ and } \operatorname{im}\eta = \operatorname{End}_{S_n}(V^{\otimes n})$$

In particular the actions commute with each other.

Let $\eta_n : U(\mathfrak{gl}_r)^{\otimes n} \longrightarrow \operatorname{End}(V^{\otimes n})$ be the representation afforded by $V^{\otimes n}$. By Proposition 2.2.10, $\operatorname{im}\eta_n \circ \varphi_r \subset \operatorname{End}_{\mathfrak{gl}_r}(V^{\otimes n})$, so by Theorem 2.2.11 the operators $\Omega^{(ab)}$ must coincide with operators coming from $\mathbb{C}S_n$.

Lemma 2.2.12. *On $V^{\otimes n}$, the operators $\varphi_r(t_{ab}) = \Omega^{(ab)}$ coincide with the operators $(a, b) \in S_n$.*

Proof. This is easily seen by direct calculation. So that our notation does not become too cluttered, we will only present the case $n = 2$, i.e. we show Ω acts as $(1, 2)$ on $V^{\otimes 2}$. Let $e_1, e_2, \ldots, e_n$ be the standard basis of V (so that $e_{ij} \cdot e_k = \delta_{jk} e_i$) and let $u \otimes v \in V^{\otimes 2}$, where $u = \sum_k u_k e_k$ and $v = \sum_k v_k e_k$. Then

$$\begin{aligned}
\Omega \cdot u \otimes v &= \sum_{i,j} e^{(1)}_{ij} e^{(2)}_{ji} u \otimes v \\
&= \sum_{i,j} e_{ij} u \otimes e_{ji} v \\
&= \sum_{i,j,k,l} u_k e_{ij} e_k \otimes v_l e_{ji} e_l \\
&= \sum_{i,j,k,l} \delta_{jk} \delta_{il} u_k v_l e_i \otimes e_j \\
&= \sum_{k,l} v_l e_l \otimes u_k e_k \\
&= v \otimes u.
\end{aligned}$$

□

This means one can identify the image of $G(z)$ in $\operatorname{End}(V^{\otimes n})$ with the image of the subalgebra $G^S(z)$ of $\mathbb{C}S_n$ generated by the elements

$$H^S_a(z) = \sum_{b \neq a} \frac{(a, b)}{z_a - z_b}.$$

Since it will usually be easier to make calculations in $\mathbb{C}S_n$, we will use this fact often in examples. Another way to express this property is to say the map $\eta_n \circ \varphi_r$ factorises:

$$\begin{tikzcd}
& \mathbb{C}S_n \arrow[dr, "\rho"] & \\
U(\mathfrak{t}_n) \arrow[ur, "\varphi'"] \arrow[rr, "\eta_n \circ \varphi_r"] & & \operatorname{End}(V^{\otimes n}),
\end{tikzcd}$$

where φ' is the map which sends t_{ab} to the transposition (a, b).

Remark 2.2.13. By Proposition 2.2.10, the action of S_n on $V^{\otimes}$ restricts to an action on $[V^{\otimes n}]_{\mu}^{\mathsf{sing}}$ for any partition μ of n. By the decomposition (2.2.5) this gives a construction of the irreducible module $S(\mu)$ of S_n corresponding to μ. Let us denote the Gaudin spectrum on $S(\mu)$ by π_{μ}°.

2.2.7 Ramification of the Gaudin spectrum

As remarked in Section 2.2.4, the main problem we wish to understand is how the spectrum of the $G(z)$ changes as we vary $z \in M_{0,n+1}(\mathbb{C})$. If the Gaudin spectrum $\pi_{\lambda_\bullet,\mu}^{\circ}$ is unramified then this is simply a monodromy problem which we might hope to solve by direct calculation. Unfortunately this is not the case and $\pi_{\lambda_\bullet,\mu}^{\circ}$ is often ramified in a way which is difficult to understand.

A collection of commuting linear operators on a vector space W have *simple spectrum* if the operators are diagonalisable and if W decomposes into one dimensional simultaneous eigenspaces. There are two ways in which we could pick up ramification. First, the algebra $G(z)$ could fail to be diagonalisable. Second, the algebra $G(z)$ could be diagonalisable but fail to have simple spectrum for a particular z, while having simple spectrum generically. Both of these need the dimension of the vector space we are acting on to be at least two, hence $\mu = (2,1)$ is the first time something could go wrong.

Proposition 2.2.14. *When $\mu = (2,1)$, the action of $G(z)$ on $S(\mu)$ is diagonalisable with simple spectrum for all $z \in M_{0,4}(\mathbb{C}) - \{w^+, w^-\}$, where $w^{\pm} = (0, 1, \frac{1}{2}(1 \pm i\sqrt{3}), \infty)$. In other words, the Gaudin spectrum is a double cover of $M_{0,4}(\mathbb{C})$, ramified only at the points $w^{\pm}$.*

Proof. The three dimensional vector space $\mathbb{C}\{v_1, v_2, v_3\}$ has a natural action of S_3 by permutation of the basis vectors. We can realise the representation $S(\mu)$ as the submodule spanned by $e_1 = v_2 - v_1$ and $e_2 = v_3 - v_1$. We will calculate the spectrum of $G(z)$ on $S(\mu)$ for a point

$$z = (z_1, z_2, z_3, \infty) \in M_{0,4}(\mathbb{C})$$

Using Proposition 2.2.5, we may assume $z_1 = 0, z_3 = 1$ and $z_2 = u$, where $u \in \mathbb{P}^1$ is distinct from $\{0, 1, \infty\}$. The generators of $G(z)$, their matrix representations with respect to the basis $\{e_1, e_2\}$ of $S(\mu)$ and their eigenspaces are summarised in Table 2.1.

From the table we see the algebra $G(z)$ has two linearly independent eigenspaces unless $\sqrt{1 - u + u^2} = 0$. This happens only when $u = \frac{1}{2}(1 \pm i\sqrt{3})$, a primitive 6^{th} root of unity. Moreover, for this value of u, the algebra $G(z)$ has only a single one-dimensional eigenspace and is thus not diagonalisable □

For more general μ and more general $\lambda_\bullet$, the ramification locus becomes much harder to calculate. This means it will be very difficult to understand the monodromy of the

operator	matrix representation	eigenvalues	eigenvectors
$H_1 = -(1,2) - \frac{(1,3)}{u}$	$\begin{pmatrix} \frac{1-u}{u} & \frac{1}{u} \\ 1 & \frac{u-1}{u} \end{pmatrix}$	$\pm\frac{1}{u}\omega$	$\begin{pmatrix} 1-u\pm\omega \\ u \end{pmatrix}$
$H_2 = (1,2) + \frac{(2,3)}{1-u}$	$\begin{pmatrix} -\frac{1}{u} & \frac{1}{u(u-1)} \\ \frac{1}{u-1} & \frac{1}{u} \end{pmatrix}$	$\pm\frac{1}{u(u-1)}\omega$	$\begin{pmatrix} 1-u\pm\omega \\ u \end{pmatrix}$
$H_3 = \frac{(1,3)}{u} + \frac{(2,3)}{u-1}$	$\begin{pmatrix} 1 & \frac{1}{1-u} \\ \frac{u}{1-u} & -1 \end{pmatrix}$	$\pm\frac{1}{1-u}\omega$	$\begin{pmatrix} 1-u\pm\omega \\ u \end{pmatrix}$

Table 2.1: The action of $G(z)$ on $S(2,1)$ where $\omega = \sqrt{1-u+u^2}$.

Gaudin spectrum in general. Below we preview a result which will be stated in full in Theorem 3.1.10.

Proposition 2.2.15. *The action of $G(z)$ on $L(\lambda_\bullet)^{\mathsf{sing}}_\mu$ is diagonalisable for $z \in M_{0,n+1}(\mathbb{R})$, that is, when z is an n-tuple of distinct real numbers.*

Proof. This is a consequence of Theorem 3.1.10 since $G(z)$ is naturally a subalgebra of the Bethe algebra (see Proposition 3.1.6). □

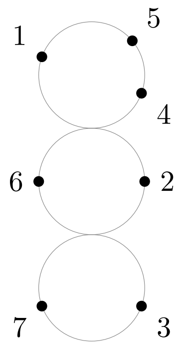
This proposition tells us that we can restrict to the real points of $M_{0,n+1}(\mathbb{C})$ and hopefully calculate the monodromy of the restricted Gaudin spectrum, this will allow us to at least understand part of the monodromy of the Gaudin spectrum. However, here we immediately run into a problem, $M_{0,n+1}(\mathbb{R})$ is isomorphic to $\{z \in \mathbb{R}^n \mid z_i \neq z_j \text{ for } i \neq j\} / \mathrm{Aff}_1^{\mathbb{R}}$, where the affine group now acts by real translations and dilations. This space is made up of connected components which are simply connected. Thus we do not expect any monodromy. The solution will be to consider a compactification of this space and we do this below. By Theorem B.1.6, birational spaces have the same monodromy.

2.3 Stable curves

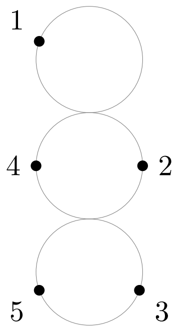
In this section we recall a compactification of the moduli space $M_{0,k}(\mathbb{C})$. This will be constructed as a moduli space of more general marked curves.

Definition 2.3.1. A *stable curve of genus zero with k marked points* is a pair (C, z), consisting of a curve C together with k points $z = (z_1, z_2, \dots, z_k)$ such that

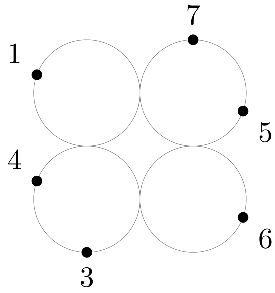
(i) the nodes of the curve are ordinary double points,



(a) This is a stable curve with 7 marked points.



(b) This is not a stable curve, it violates 2.3.1 condition (iv).



(c) This is also not a stable curve, it violates 2.3.1 condition (ii).

Figure 2.1: An Example and non-examples of stable curves

(ii) the number of nodes is one less than the number of irreducible components,

(iii) the k marked points are distinct and do not coincide with the nodes, and

(iv) each irreducible component contains at least three special points, where a special point means either a marked point or a node.

A morphism of stable curves $\phi\colon(C, z_1, z_2, \ldots, z_k) \longrightarrow (C', z_1', z_2', \ldots, z_k')$ is a morphism $\phi\colon C \longrightarrow C'$ such that $\phi(z_i) = z_i'$.

Remark 2.3.2. Note condition (ii) is equivalent to specifying that the graph of irreducible components has no closed circuits. Condition (iv) means that the curve has no non-trivial automorphims which fix the special points, hence the name *stable*. These conditions also imply that a stable curve with k marked points can have at most $k-2$ irreducible components. We will generally represent stable curves by diagrams of the form in Figure 2.1.

2.3.1 The moduli space of stable curves

It is clear from the definition that an irreducible, genus zero curve with k marked points is an example of a stable curve, and indeed the moduli space of stable curves can be viewed as a compactification of $M_{0,k}(\mathbb{C})$.

Theorem 2.3.3 ([Knu83]). *For $k \geq 3$ there exists a fine moduli space $\overline{M}_{0,k}(\mathbb{C})$ of stable, genus zero curves with k marked points.*

Example 2.3.4. When $k = 3$, Remark 2.3.2 tells us that there can be at most 1 connected component. In fact, by Proposition 2.2.5 any two stable curves $(\mathbb{P}^1, z_1, z_2, z_3)$

and $(\mathbb{P}^1, z_1', z_2', z_3')$ are isomorphic. Hence there is only a single stable curve with three marked points and thus $\overline{M}_{0,k}(\mathbb{C}) = M_{0,k}(\mathbb{C})$ is a point.

Remark 2.3.5. As part of Theorem 2.3.3, the moduli space $\overline{M}_{0,k}(\mathbb{C})$ comes packaged with a universal family $\overline{U}_{0,k}$, a scheme over $\overline{M}_{0,k}(\mathbb{C})$ with k sections $\sigma_i : \overline{M}_{0,k}(\mathbb{C}) \longrightarrow \overline{U}_{0,k}$, for $i = 1, 2, \ldots, k$. The geometric fibre over a stable curve $C \in \overline{M}_{0,k}(\mathbb{C})$ is a curve isomorphic to C, with $\sigma_i(C)$ being the k marked points.

We will not recall the construction of $\overline{M}_{0,k}(\mathbb{C})$ here, and refer the reader to [KV07, Chapter 2]. A description of the topology of $\overline{M}_{0,k}(\mathbb{C})$ is given by considering collisions of special points. More precisely, suppose C is a stable curve, let $D \cong \mathbb{P}^1$ be an irreducible component of C with a marked point $p \in D$ and a special point $x \in D$. Choose a path $\rho : [0, 1) \longrightarrow D$, such that $\rho(u)$ is not a special point for any $u \in (0, 1)$, $\rho(0) = p$, and $\lim_{u \to 1} \rho(u) = x$. Let $\Gamma(u) \in \overline{M}_{0,k}(\mathbb{C})$ be the stable curve, isomorphic (as varieties) to C, with the same marked points except $\rho(u)$ replacing the marked point at p (so $\Gamma(0) = C$). This defines a path $\Gamma : [0, 1) \longrightarrow \overline{M}_{0,k}(\mathbb{C})$. There are several situations which can arise:

(i) There are exactly 3 special points on D. In this case, by Proposition 2.2.5, there is an isomorphism of stable curves $\Gamma(u) \cong C$, for all $u \in [0, 1)$. In particular, Γ is constant, so $\lim_{u \to 1} \Gamma(u) = C$.

(ii) The point $x \in D$ is a marked point and D has more than 3 special points. In this case $\lim_{u \to 1} \Gamma(u)$ is a stable curve with one extra irreducible component than C. The extra component is attached to D at x and it contains exactly two marked points.

(iii) The point $x \in D$ is a node and D has more than 3 special points. In this case $\lim_{u \to 1} \Gamma(u)$ is again a stable curve with one extra irreducible component then C. To construct this stable curve, we blow up C at x, remove the marked point at $p \in D$ and add a marked point on the new irreducible component resulting from the blow up.

This procedure is shown in Figure 2.2.

Example 2.3.6. Consider the case $k = 4$. The open set $M_{0,4}(\mathbb{C})$ is isomorphic to $\mathbb{P}^1 - \{0, 1, \infty\}$. One choice for this isomorphism is given by identifying the point $u \in M_{0,4}(\mathbb{C}) \cong \mathbb{P}^1 - \{0, 1, \infty\}$ with the stable curve with one irreducible component and marked points at $(0, u, 1, \infty)$. There are exactly 3 stable curves with 2 irreducible components, the curve with marked points $(1, 2)$ on the same component, the curve with marked points $(1, 3)$ on a common component and the curve with marked points $(1, 4)$ on a common component. We obtain the picture in Figure 2.3.

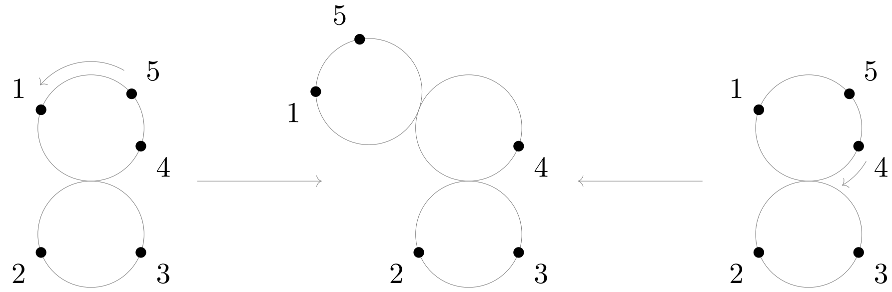


Figure 2.2: Special points colliding in two different ways to the same stable curve

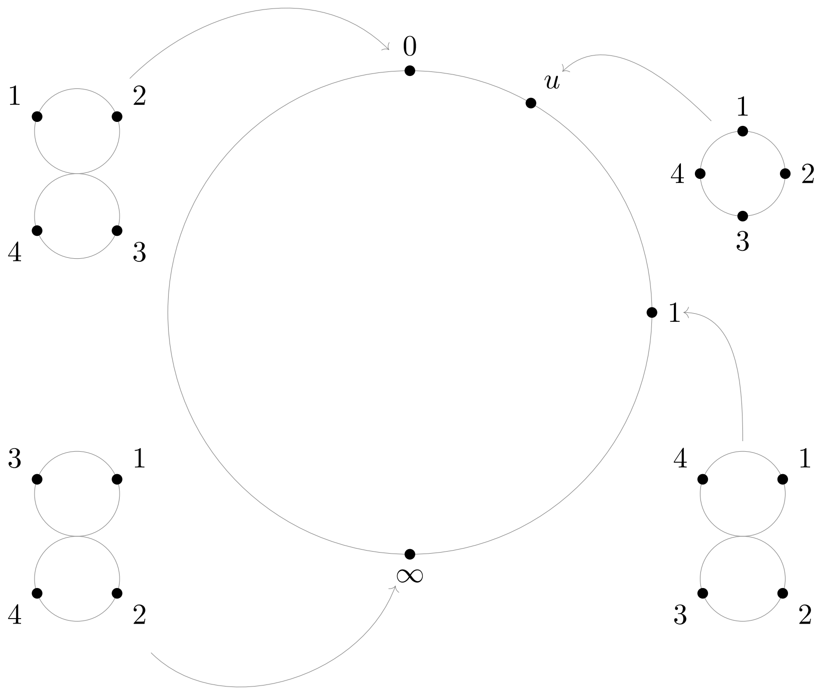
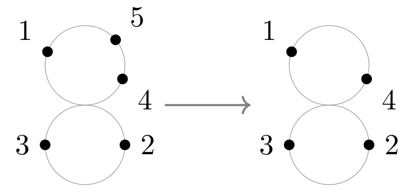


Figure 2.3: The moduli space $\overline{M}_{0,4}(\mathbb{C})$

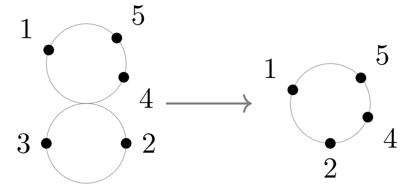
Actually, Figure 2.3 is a picture of only the real points. In this case the space $\overline{M}_{0,4}(\mathbb{R})$ is made up of three 1-simplices with boundaries are the points marked as $0, 1$, and ∞. The real points, $\overline{M}_{0,k}(\mathbb{R})$, always admit a combinatorial description as a CW-complex and we will revisit this in Section 4.5.

2.3.2 Contraction and stabilisation

Given a stable curve with k marked points, $C \in \overline{M}_{0,k}(\mathbb{C})$, and $i \in [k]$, we can produce a stable curve with $k-1$ marked points by forgetting the i^{th} marked point. Any irreducible component that is no longer stable (has less than 3 special points) is contracted. See Figure 2.4 for an illustration of this operation. The resulting curve is the *contraction* of

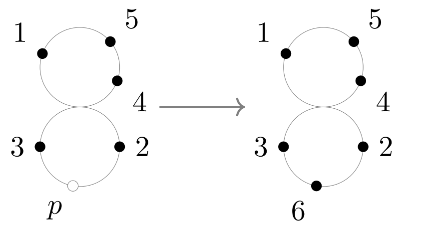


(a) Contraction at 5

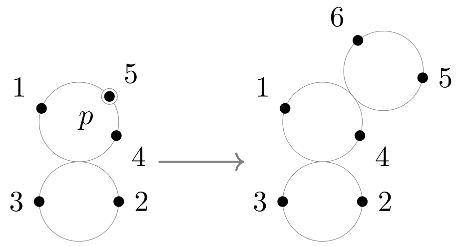


(b) Contraction at 3

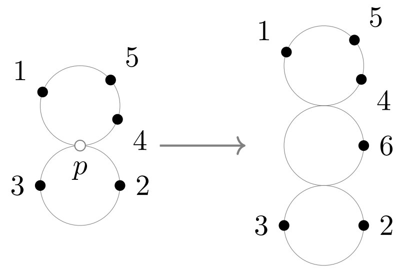
Figure 2.4: Contraction of stable curves



(a) Stabilisation at a non-special point



(b) Stabilisation at a marked point



(c) Stabilisation at a node

Figure 2.5: Stabilisation of stable curves

C at i.

There is also an operation in the opposite direction called *stabilisation*. Given a stable curve with k marked points, $C \in \overline{M}_{0,k}(\mathbb{C})$ and a point $p \in C$ (which we can think of as a point $p \in \overline{U}_{0,k}(\mathbb{C})$ in the fibre of C) we construct a stable curve with $k+1$ marked points in the following way. If p is not a special point we simply add a marked point at p labelled $k+1$. If $p \in C$ is a special point we add an irreducible component at this special point and place a marked point p on this new irreducible component. This defines the stabilisation of C at p *up to projective equivalence*. Again we illustrate this with examples in Figure 2.5.

Definition 2.3.7. The operations of contraction and stabilisation define the inverse maps

$$c_{k+1} : \overline{M}_{0,k+1}(\mathbb{C}) \longrightarrow \overline{M}_{0,k}(\mathbb{C}) \quad \text{and} \quad s_k : \overline{U}_{0,k}(\mathbb{C}) \longrightarrow \overline{M}_{0,k+1}(\mathbb{C}).$$

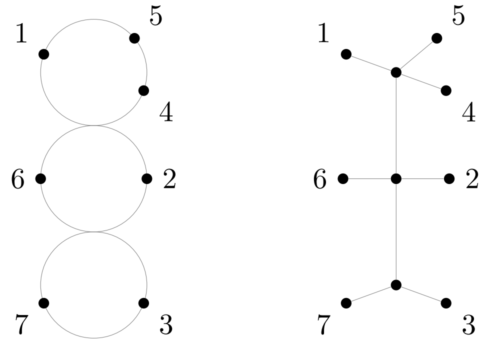


Figure 2.6: A stable curve and its corresponding dual graph.

Proposition 2.3.8 ([Knu83, Theorem 2.4]). *The maps c_{k+1} and s_k are morphisms of varieties, moreover, c_{k+1} is flat and proper and s_k is an isomorphism over $\overline{M}_{0,k}(\mathbb{C})$. That is, as families of stable pointed curves we have an isomorphism*

$$\begin{tikzcd}
\overline{U}_{0,k} \arrow[rr, "\sim", "s_k"'] \arrow[dr] & & \overline{M}_{0,k+1}(\mathbb{C}) \arrow[dl, "c_{k+1}"] \\
& \overline{M}_{0,k}(\mathbb{C}) &
\end{tikzcd}.$$

2.3.3 Stratification of $\overline{M}_{0,k}(\mathbb{C})$

Proposition 2.3.8 shows that the moduli space $\overline{M}_{0,k}(\mathbb{C})$ is constructed in an inductive way. There is a stratification of $\overline{M}_{0,k}(\mathbb{C})$ which reflects this inductive structure. Let $M_i \subset \overline{M}_{0,k}(\mathbb{C})$ be the closed subvariety of stable curves with at least i irreducible components. Let $U_i = M_i - M_{i+1}$. Thus $M_1 = \overline{M}_{0,k}(\mathbb{C})$ and $U_1 = M_{0,k}(\mathbb{C})$. This produces a stratification

$$\overline{M}_{0,k}(\mathbb{C}) = M_1 \supset M_2 \supset \ldots \supset M_{k-2} \supset M_{k-1} = \emptyset.$$

The subvariety M_i is $k-i-2$-dimensional. In particular, M_{k-2} is a finite set of points.

Given a stable curve $C \in \overline{M}_{0,k}(\mathbb{C})$ we can associate a labelled tree, the *dual graph* of C. For each irreducible component of C we have a vertex and for each special point, an edge. The edges corresponding to node points, connect the corresponding irreducible components. For each $j \in [k]$ we add a leaf connected to the vertex representing the irreducible component containing the point marked j.

The irreducible components of M_i are indexed by trees with leaves labelled by $[k]$ and with i internal vertices that each have at least three edges incident. Thus the points of M_{k-2}, i.e. those stable curves with only three special points on each component, are labelled by trees with $k-2$ internal vertices.

Lemma 2.3.9. *A tree T with k leaves and $k-2$ internal vertices, all with minimum degree 3 must be trivalent.*

Proof. This is a consequence of the *degree sum formula*,

$$2\#E = \sum_{v \in T} \deg(v),$$

where $\#E$ is the number of edges in T and the sum ranges over the vertices (including leaves) of T. The total number of nodes in T is $k + (k-2) = 2k-2$. Thus the total number of edges is $\#E = 2k-3$. Since every leaf has degree exactly one, the sum of the degrees of the internal vertices is

$$\sum_{v \in \mathrm{int}(T)} \deg(v) = 2\#E - k = 3k - 6 = 3(k-2).$$

Thus each internal vertex has degree exactly three. □

2.3.4 Trees and bracketings

It will be useful to have another way of indexing the irreducible components of the strata M_i. We will also use this description to describe the real points $\overline{M}_{0,k}(\mathbb{R})$ as a CW-complex in Section 4.5.

Definition 2.3.10. An *unordered i-bracketing* of the set $[n]$ is an arrangement of these numbers with i pairs of balanced brackets. We require that there is a pair of brackets surrounding the entire expression. An unordered bracketing is *nontrivial* if all pairs of brackets contain at least two subunits (a letter, or another pair of balanced brackets). Two unordered bracketings are equivalent if we can obtain one from another by permuting the contents of one or more pairs of balanced brackets.

An *ordered i-bracketing* (or simply an i-bracketing) is the same thing as an unordered i-bracketing however we only consider two equivalent if they differ by reversing the contents of one or more pairs of balanced brackets.

For example (13(42(57))6) is a nontrivial 3-bracketing of the set [7], but the bracketings (3(4)(12)) and ((153)(24)) are trivial. The bracketings (1(432)5) and (5(432)1) are equivalent as both ordered and unordered bracketings but the bracketing (1(423)5) is only equivalent to (1(432)5) as an unordered bracketing.

Proposition 2.3.11. *There is a bijection of sets*

$$\left\{\begin{matrix}\text{rooted trees with leaves}\\ \text{labelled by } 1,2,\ldots,n\\ \text{and } i \text{ internal vertices}\end{matrix}\right\} \longleftrightarrow \{\text{unordered } i\text{-bracketings}\},$$

where both sets are considered up to equivalence. Furthermore if we consider only those rooted trees where every internal vertex has degree at least three and the root has degree at least two, this corresponds to nontrivial unordered i-bracketings.

Proof. Inductively on i we will construct an i-bracketing from a rooted tree. When $i = 1$, the only unordered bracketing is $(12 \cdots n)$ and there is only a single rooted tree with a single internal vertex. Now suppose that $i > 1$. If we remove the root vertex and its incident edges we are left with a collection of rooted trees, each of which have fewer than i internal vertices. So by induction, each of these subtrees has a corresponding bracketing. Concatenating these bracketings and surrounding them by a single pair of brackets produces an unordered i-bracketing. The inverse map is constructed analogously.

If the root of the tree we started with had degree at least two, then the top level bracket has at least two subunits. The subtrees after removing the root and its incident edges have the same property. Thus by induction the claim is proved. □

Example 2.3.12. Consider the tree in Figure 2.6 as a rooted tree with root the vertex neighbouring the leaf marked 6. The corresponding unordered 2-bracketing is $(6(154)2(73))$.

As discussed in Section 2.3.3, each stable curve $C \in \overline{M}_{0,k}(\mathbb{C})$ has a corresponding dual graph. We can turn this into a rooted tree by taking as root the vertex with leaf labelled by k and deleting this leaf. These rooted trees always have the property that the minimum degree of an internal vertex is three.

Corollary 2.3.13. *The irreducible components of M_i are labelled by unordered i-bracketings of $\{1, 2, \dots, k-1\}$.*

Corollary 2.3.13 implies the points of M_{k-2} are given by nontrivial bracketings with the maximum number of brackets, i.e. if we consider $1, 2, \dots, k-1$ as formal symbols which can be multiplied, this is exactly the ways in which to multiply these unambiguously, without appealing to associativity. We will use this notation for these *maximally degenerate* stable curves freely throughout the rest of this thesis.

Example 2.3.14. When $k = 4$,

$$M_2 = \{((12)3), (1(23)), ((13)2)\}.$$

2.4 Gaudin subalgebras

We are now ready to recall the theorem of Aguirre, Felder and Veselov, extending the definition of $G(z)$ to points $z \in \overline{M}_{0,n+1}(\mathbb{C})$. The real points $\overline{M}_{0,n+1}(\mathbb{R})$ have nontrivial fundamental group so we anticipate that we can restrict the spectrum of this family of algebras to $\overline{M}_{0,n+1}(\mathbb{R})$ and calculate some monodromy. In Section 2.4.4 we provide an example which shows the family of algebras obtained is not flat and thus the spectrum is ramified at a certain point in the boundary of $\overline{M}_{0,n+1}(\mathbb{R})$.

Since the relations in the Konho-Drinfeld Lie algebra $\mathfrak{t}_n$, from Definition 2.2.1 are homogeneous, $\mathfrak{t}_n$ is naturally graded. Let $\mathfrak{t}_n^1$ be the degree 1 graded piece, the span of the generators t_{ij}.

Definition 2.4.1. A *Gaudin subalgebra* is an abelian Lie subalgebra of $\mathfrak{t}_n$ contained in $\mathfrak{t}_n^1$, of maximal dimension.

Proposition 2.4.2 ([AFV11, Corollary 2.13])**.** *An abelian Lie subalgebra of $\mathfrak{t}_n^1$ has dimension at most $n-1$.*

Example 2.4.3. Let L be the Lie-subalgebra generated by the elements

$$\begin{aligned} L_2 &= t_{12}, \\ L_3 &= t_{13} + t_{23}, \\ &\vdots \\ L_n &= t_{1n} + t_{2n} + \ldots + t_{(n-1)n}. \end{aligned}$$

Then L is clearly of dimension $n-1$ and a short calculation shows it is abelian, and thus a Gaudin subalgebra. The map $\varphi_{S_n} : U(\mathfrak{t}_N) \longrightarrow \mathbb{C}S_n$ identifies L with the *Jucys-Murphy elements*.

Example 2.4.4. The algebras $G(z)$ from Definition 2.2.2 are all of dimension $n-1$ by Lemma 2.2.3 and thus are Gaudin subalgebras.

The algebras $G(z)$ will turn out to be the generic example of Gaudin subalgebras in the sense that all Gaudin subalgebras appear naturally as limits of $G(z)$ in the next section.

2.4.1 Moduli of Gaudin subalgebras

Example 2.4.4 motivates the following theorem of Aguirre, Felder and Veselov in [AFV11], describing the moduli of the Gaudin subalgebras.

Theorem 2.4.5 ([AFV11, Theorem 2.5])**.** *The subset of* $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$*, the Grassmanian of $(n-1)$-planes in $\mathfrak{t}_n^1$, parametrising the Gaudin subalgebras defines a smooth subvariety isomorphic to $\overline{M}_{0,n+1}(\mathbb{C})$.*

It is possible, in principle, to calculate the Gaudin subalgebra corresponding to a point in $\overline{M}_{0,n+1}(\mathbb{C})$ by taking the limit of subalgebras of the form $G(z)$. The following detailed example illustrates how this works for $n = 4$, in general a similar procedure works.

Example 2.4.6. Let $n = 4$. We will calculate the Gaudin subalgebra corresponding to the point $(((12)3)4) \in M_{0,5}(\mathbb{C})$. The generators H_1, H_2, H_3, H_4 of $G(z)$ are given by the equations

$$\begin{aligned}
H_1 &= \frac{t_{12}}{z_1 - z_2} + \frac{t_{13}}{z_1 - z_3} + \frac{t_{14}}{z_1 - z_4}, \\
H_2 &= -\frac{t_{12}}{z_1 - z_2} + \frac{t_{23}}{z_2 - z_3} + \frac{t_{24}}{z_2 - z_4}, \\
H_3 &= -\frac{t_{13}}{z_1 - z_3} - \frac{t_{23}}{z_2 - z_3} + \frac{t_{34}}{z_3 - z_4}, \\
H_4 &= -\frac{t_{14}}{z_1 - z_4} - \frac{t_{24}}{z_2 - z_4} - \frac{t_{34}}{z_3 - z_4}.
\end{aligned}$$

Let $z = (u^3, u^2, u, 1, \infty)$ for some small $u \in \mathbb{R}$. We will calculate the limit as u goes to 0. By the discussion in Section 2.3.1 this will give the Gaudin algebra at the point $(((12)3)4)$. Thus

$$\begin{aligned}
H_1 &= \frac{t_{12}}{u^3 - u^2} + \frac{t_{13}}{u^3 - u} + \frac{t_{14}}{u^3 - 1}, \\
H_2 &= -\frac{t_{12}}{u^3 - u^2} + \frac{t_{23}}{u^2 - u} + \frac{t_{24}}{u^2 - 1}, \\
H_3 &= -\frac{t_{13}}{u^3 - u} - \frac{t_{23}}{u^2 - u} + \frac{t_{34}}{u - 1}, \\
H_4 &= -\frac{t_{14}}{u^3 - 1} - \frac{t_{24}}{u^2 - 1} - \frac{t_{34}}{u - 1}.
\end{aligned}$$

We will take the limit of the rescaled operators

$$\begin{aligned}
H_1' &= \frac{1}{u}(u^3 - u^2)(u^3 - u)(u^3 - 1)H_1, \\
H_2' &= -\frac{1}{u}(u^3 - u^2)(u^2 - u)(u^2 - 1)H_2, \\
H_3' &= -\frac{1}{u}(u^3 - u)(u^2 - u)(u - 1)H_3, \\
H_4' &= -(u^3 - 1)(u^2 - 1)(u - 1)H_4,
\end{aligned}$$

which are

$$\begin{aligned}
H_1' &= (u^2 - 1)(u^3 - 1)t_{12} + (u^2 - u)(u^3 - 1)t_{13} + (u^2 - u)(u^3 - u)t_{14}, \\
H_2' &= (u - 1)(u^2 - 1)t_{12} - (u^2 - u)(u^2 - 1)t_{23} - (u^2 - u)(u^2 - u)t_{24}, \\
H_3' &= (u - 1)^2 t_{13} + (u^2 - 1)(u - 1)t_{23} - (u^2 - 1)(u^2 - u)t_{34}, \\
H_3' &= (u^2 - 1)(u - 1)t_{14} + (u^3 - 1)(u - 1)t_{24} + (u^3 - 1)(u^2 - 1)t_{34}.
\end{aligned}$$

Now we are able to take a well defined limit as u goes to zero. The operators become

$$\begin{aligned}
\lim_{u \to 0} H_1' &= t_{12}, \\
\lim_{u \to 0} H_2' &= t_{12},
\end{aligned}$$

$$\lim_{u \to 0} H_3' = t_{13} + t_{23},$$
$$\lim_{u \to 0} H_3' = t_{14} + t_{24} + t_{34}.$$

These operators have as their image in $\mathbb{C}S_4$ the *Jucys-Murphy operators*. In fact this is true in general.

Using the action of the affine group, the limit of $(u^3, u^2, u, 1, \infty) \in M_{0,5}(\mathbb{C})$ as u tends to zero is the same as the limit of $(u, u^2, u^3, u^4, \infty)$ as u tends to ∞. In fact the limit of the Gaudin Hamiltonians in the second parameterisation is easier to evaluate as shown in the proof of the following proposition. In Example 2.4.6 we chose the more difficult parameterisation in order to illustrate this point as we will use both parameterisations later.

Let $f(u)$ and $g(u)$ be functions of a real variable u. Recall that $f = O(g)$ if

$$\lim_{u \to \infty} \left| \frac{f(u)}{g(u)} \right| < \infty$$

and $f = o(g)$ if

$$\lim_{u \to \infty} \left| \frac{f(u)}{g(u)} \right| = 0.$$

The following result is well known.

Proposition 2.4.7. *Let $z = (z_1, z_2, \ldots, z_n, \infty)$ where $z_i = z_i(u)$ depends on a real variable u such that $\lim_{u \to \infty} z_i(u) = \infty$ and $z_i = o(z_{i+1})$. The limit of the algebra $G(z)$ as u goes to ∞ is spanned by the Jucys-Murphy elements*

$$L_a = t_{1a} + t_{2a} + \ldots + t_{(a-1)a},$$

for $a = 2, 3, \ldots, n$. In particular, the Gaudin subalgebra corresponding to the point $(((12)3) \cdots n) \in \overline{M}_{0,n+1}(\mathbb{C})$ is $\mathbb{C}\{L_2, L_3, \ldots, L_n\}$.

Proof. We take the limit of the rescaled operators

$$z_a H_a(z) = \sum_{b \neq a} \frac{z_a}{z_a - z_b} t_{ab}.$$

Since

$$\lim_{u \to \infty} \frac{z_a}{z_a - z_b} = \begin{cases} 1 & \text{if } b < a, \\ 0 & \text{if } b > a, \end{cases}$$

we have that

$$\lim_{u \to \infty} z_a H_a(z) = \sum_{b < a} t_{ab} = L_a. \qquad \square$$

2.4.2 The Gaudin spectrum compactified

In a similar way to Section 2.2.4 we can construct a ramified covering of $\overline{M}_{0,n+1}(\mathbb{C})$. Let $\overline{G}$ be the restriction of the tautological bundle on $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$ to the subvariety $\overline{M}_{0,n+1}(\mathbb{C})$ cut out by the Gaudin subalgebras (as provided by Theorem 2.4.5). Thus $\overline{G}$ is a bundle of Lie algebras on $\overline{M}_{0,n+1}(\mathbb{C})$, the fibre over a point being the corresponding Gaudin subalgebra. The restriction to $M_{0,n+1}(\mathbb{C}) \subset \overline{M}_{0,n+1}(\mathbb{C})$ is G° from Section 2.2.4. We denote the sheaf of sections of $\overline{G}$ by $\overline{\mathcal{G}}$, and for a representation W of $U(\mathfrak{gl}_r)^{\otimes n}$ we can apply the same construction as in Section 2.2.4 to construct

$$\operatorname{Sym} \overline{\mathcal{G}} \longrightarrow \mathcal{E}nd(W).$$

The image is denoted $\overline{\mathcal{G}}_W$. The associated spectrum

$$\overline{\pi}_W : \operatorname{Spec} \overline{\mathcal{G}}_W \longrightarrow \overline{M}_{0,n+1}(\mathbb{C})$$

is a finite map and $\operatorname{Spec} \overline{\mathcal{G}}_W$ compactifies $\operatorname{Spec} \mathcal{G}_W^\circ$. We call the map $\overline{\pi}_W$ the *compactified Gaudin spectrum* associated to W. We will be primarily interested in the special case $W = L(\lambda_\bullet)_\mu^{\mathsf{sing}}$, where we denote the Gaudin spectrum $\overline{\pi}_{\lambda_\bullet,\mu}$.

2.4.3 An example of monodromy in the compactification

We saw in Section 2.2.7 that the Gaudin spectrum $\pi^\circ_{\lambda_\bullet,\mu}$ and hence the compactified Gaudin spectrum $\overline{\pi}_{\lambda_\bullet,\mu}$, may be ramified at complex points. Motivated by Proposition 2.2.15 we would like to restrict to the real points of $M_{0,n+1}(\mathbb{C})$ and calculate some monodromy here. The problem is that $M_{0,n+1}(\mathbb{R})$ has connected components which are all simply connected. Instead we have compactified, obtaining a map $\overline{\pi}_{\lambda_\bullet,\mu}$ whose Galois theory we know is the same as $\pi^\circ_{\lambda_\bullet,\mu}$. Our strategy is to restrict to $\overline{M}_{0,n+1}(\mathbb{R})$ in order to calculate the monodromy (though there may be extra monodromy over the complex points).

As a first example to illustrate this strategy we return to situation of Section 2.2.7, namely we consider the action of the Gaudin subalgebras for $n = 3$ on the irreducible S_3 module $S(\mu)$ where $\mu = (2, 1)$. As shown in Example 2.3.6, there are 3 extra points in $\overline{M}_{0,4}(\mathbb{R})$ compared to $M_{0,4}(\mathbb{R})$ which correspond to the bracketings

$$((12)3), (1(23)), \text{ and } ((13)2).$$

Lemma 2.4.8. *The compactified Gaudin spectrum $\overline{\pi}_{(2,1)}$ is unramified over $\overline{M}_{0,4}(\mathbb{R})$.*

Proof. In Section 2.2.7 it was shown that $\overline{\pi}_{(2,1)}$ is unramified over the dense open subset $M_{0,4}(\mathbb{R})$. By Proposition 2.4.7 the Gaudin subalgebra corresponding to $((12)3)$ is given by the Jucys-Murphy elements, $\mathbb{C}\{L_2, L_3\}$, which act by the matrix

$$L_2, -L_3 \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \tag{2.4.1}$$

with respect to the basis $\{e_2, e_3\}$ of $S(\mu)$ given in Section 2.2.7. This matrix has two distinct eigenvalues ± 1. It is thus diagonalisable with simple spectrum. This also means $\overline{\pi}_{(2,1)}$ is unramified over this point.

By taking limits in an appropriate fashion, we also see that the Gaudin subalgebras at the remaining two points are

$$G(1(23)) = \mathbb{C}\{t_{12} + t_{13}, t_{23}\} \quad \text{and } G((13)2) = \mathbb{C}\{t_{13}, t_{12} + t_{23}\}.$$

The image of these algebras in $\mathbb{C}S_3$ are both conjugate to $\mathbb{C}\{L_2, L_3\}$ and thus also have simple spectrum on $S(\mu)$. Hence $\overline{\pi}_{(2,1)}$ is an unramified double cover of $\overline{M}_{0,4}(\mathbb{R})$. $\square$

We will now calculate the monodromy action of $\pi_1(\overline{M}_{0,4}(\mathbb{R})) = \mathbb{Z}$. We take as our basepoint $((12)3)$ and as generator, γ, the loop traversing $\overline{M}_{0,4}(\mathbb{R})$ in the anticlockwise direction of the depiction of Figure 2.3. We identify $\overline{M}_{0,4}(\mathbb{R})$ with $\mathbb{P}^1(\mathbb{R})$ by the rational map

$$(0, u, 1, \infty) \mapsto u,$$

so our basepoint is $u = 0$. If $z = (0, u, 1, \infty)$ we write $G(u) = G(z)$. The spectrum of $G((12)3) = \mathbb{C}\{L_2, L_3\}$ is two points, given by the functionals $\lambda_\pm : G((12)3) \longrightarrow \mathbb{C}$, where $\lambda_\pm(L_2) = \mp 1$ and $\lambda_\pm(L_3) = \pm 1$ (which are just the eigenvalues of L_2 and L_3 on the eigenspaces of $S(\mu)$).

Similarly, if $\omega = \sqrt{1 - u + u^2}$, the spectrum of $G(u)$ is given by the functionals $\lambda_{\pm u} : G(u) \longrightarrow \mathbb{C}$, where

$$\lambda_{\pm u}(H_1) = \pm\frac{1}{u}\omega, \quad \lambda_{\pm u}(H_2) = \pm\frac{1}{u(u-1)}\omega, \quad \text{and } \lambda_{\pm u}(H_3) = \pm\frac{1}{1-u}\omega.$$

Recall from Table 2.1, these are the eigenvalues of H_1, H_2 and H_3. We now have enough information to calculate the monodromy action. The reader may wish to consult Figure 2.7, which shows pictorially the monodromy. It will be convenient below to recall the following facts.

$$\lim_{u\to 0}(-uH_1) = L_2 \tag{2.4.2}$$

$$\lim_{u\to 0}(uH_2) = L_2 \tag{2.4.3}$$

$$\lim_{u\to 0}(H_3) = L_3. \tag{2.4.4}$$

Claim 1: When $u \in (0, 1)$ we have $\lim_{u\to 0^+} \lambda_{\pm u} = \lambda_\pm$. By definition

$$\lambda_{\pm u}(-uH_1) = \mp\omega, \quad \text{and } \lambda_{\pm u}(uH_2) = \mp\frac{1}{1-u}\omega.$$

Since $u \in (0, 1)$, we have $\lim_{u\to 0^+} \omega = 1$. Thus we have $\lim_{u\to 0^+} \lambda_{\pm u} = \lambda_\pm$.

Claim 2: When $u \in (-\infty, 0)$ we have $\lim_{u\to 0^-} \lambda_{\pm u} = \lambda_{\pm}$. This is proven in an entirely analogous way.

Claim 3: When $u \in (0,1)$ and $v \in (1,\infty)$, we have

$$\lim_{u\to 1^-} \lambda_{\pm u} = \lim_{v\to 1^+} \lambda_{\pm v}.$$

To see this first note that $\lim_{u\to 1^+} \omega = \lim_{u\to 1^-} \omega = 1$. Since

$$\begin{aligned} \lim_{u\to 1^-} H_1(u) &= \lim_{v\to 1^+} H_1(v), \\ \lim_{u\to 1^-} (u-1)H_2(u) &= \lim_{v\to 1^+} (v-1)H_2(v), \\ \lim_{u\to 1^-} (1-u)H_3(u) &= \lim_{v\to 1^+} (1-v)H_3(v), \end{aligned}$$

we can just compare the value of $\lambda_{\pm u}$ and $\lambda_{\pm v}$ at these values in the limit. For example

$$\lim_{u\to 1^-} \lambda_{\pm u}((u-1)H_2(u)) = \lim_{u\to 1^-} \pm\frac{1}{u}\omega = \pm 1 = \lim_{v\to 1^+} \lambda_{\pm v}((v-1)H_2(v)).$$

Hence the claim is proved.

Claim 4: When $u \in (1,\infty)$ and $v \in (-\infty, 0)$, we have

$$\lim_{u\to\infty} \lambda_{\pm u} = \lim_{v\to -\infty} \lambda_{\mp v}$$

(note the inversion of signs). Similarly to above we can see this by noting that

$$\begin{aligned} \lim_{u\to\infty} H_1(u) &= \lim_{v\to -\infty} H_1(v), \\ \lim_{u\to\infty} uH_2(u) &= \lim_{v\to -\infty} vH_2(v), \\ \lim_{u\to\infty} H_3(u) &= \lim_{v\to -\infty} H_3(v), \end{aligned}$$

Now as above we can compare the values of $\lambda_{\pm u}$ in the limit. For example

$$\lim_{u\to\infty} \lambda_{\pm u}(uH_2(u)) = \lim_{u\to\infty} \pm\frac{1}{u-1}\omega = \pm 1.$$

However now we need to slightly more careful for the other side,

$$\lim_{v\to -\infty} \lambda_{\pm v}(vH_2(v)) = \lim_{v\to -\infty} \pm\frac{1}{v-1}\omega = \mp 1.$$

By checking the limits on the remaining generators the claim is proved.

Putting these claims together, allows us to track the spectrum of $G(u)$ as we vary u. Figure 2.7 displays these claims pictorially. We have proven the following.

Proposition 2.4.9. *Let γ be the generator of the group $\pi_1(\overline{M}_{0,4}(\mathbb{R})) = \mathbb{Z}$ defined above. The monodromy action on the fibre of $\overline{\pi}_{(2,1)}$ over $((12)3)$ is given by $\gamma \cdot \lambda_{\pm} = \lambda_{\mp}$.*

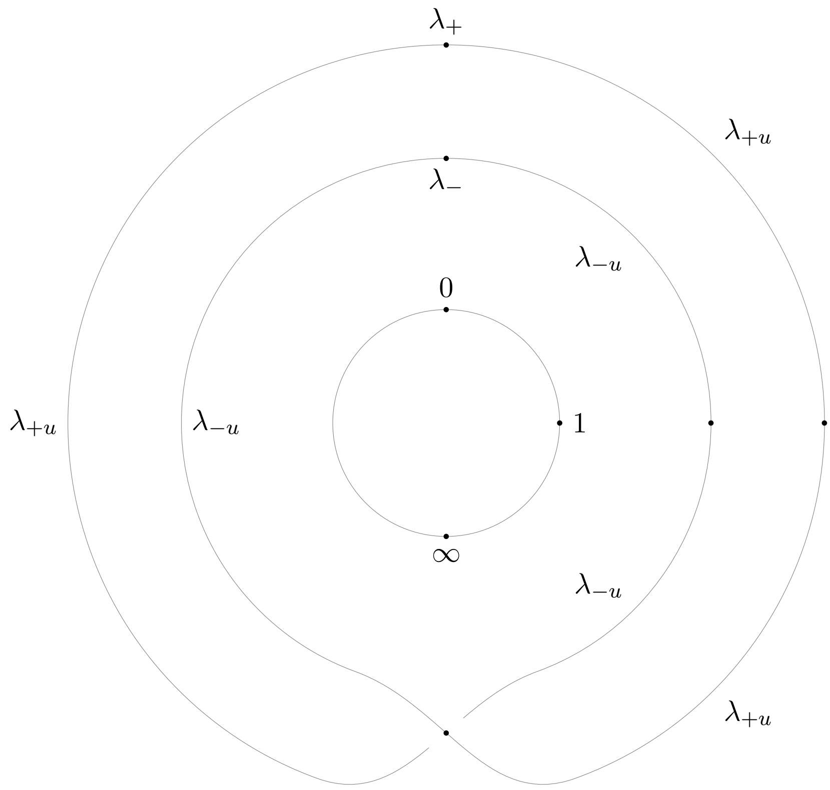


Figure 2.7: The double cover and the monodromy for $\mu = (2,1)$

Our strategy to reduce to the subspace $\overline{M}_{0,n+1}(\mathbb{R})$ has worked in the above special case, moreover combining with the results in Section 2.2.7 this provides a complete description of the monodromy of the Gaudin spectrum over $\overline{M}_{0,n+1}(\mathbb{C})$ for $n = 3$, $\lambda_\bullet = (\Box^3)$ and $\mu = (2,1)$. In the next section we demonstrate that we must revise this strategy for more general n.

2.4.4 Ramification of the compactified Gaudin spectrum

Motivated by Theorem 2.4.5, Propositions 2.2.15 and 2.4.9, we could hope that the compactified Gaudin spectrum, $\overline{\pi}_{\lambda_\bullet,\mu}$, is unramified over real points. Unfortunately this is not true, as we will demonstrate this in this section. However hope is not lost, as it turns out that over real points, if $\overline{\pi}_{\lambda_\bullet,\mu}$ happens to be ramified, it is because the corresponding Gaudin subalgebra does not have simple spectrum on $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$, not because it is not diagonalisable. This means we will be able to adapt our previous strategy of restricting to $\overline{M}_{0,n+1}(\mathbb{R})$, provided we add extra operators which are diagonalisable and commute with the Gaudin Hamiltonians. These are the *higher Gaudin Hamiltonians* and will be studied in Section 3.1.

In this section we will present an example of a Gaudin subalgebra with non-simple

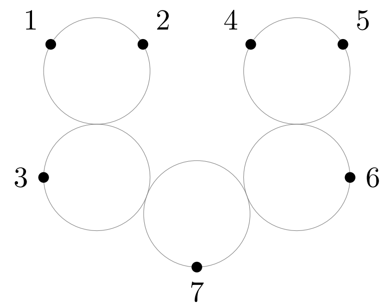


Figure 2.8: The point $((12)3)((45)6) \in \overline{M}_{0,7}$

spectrum, the spectrum will consist of a number of 1 and 2-dimensional simultaneous eigenspaces. In Appendix A we then find extra commuting operators which decompose the 2-dimensional eigenspaces. These extra operators arise as degenerations of higher Hamiltonians. Our example is for $n = 6$, and we will consider the action on the irreducible $\mathbb{C}S_6$ modules $S(\mu)$ for $\mu = (3, 2, 1)$. The point over which $\overline{\pi}_\mu$ turns out to be ramified is the point of $\overline{M}_{0,7}(\mathbb{C})$ determined by the stable curve pictured in Figure 2.8. This is the point $z_{NS} \in \overline{M}_{0,7}(\mathbb{C})$ labelled by $(((12)3)((45)6))$.

Consider the point $z = (0, u^5, u^4, u, u^3 + u, u^2 + u, \infty) \in M_{0,7}(\mathbb{C})$. The limit as $u \to 0$ of this point is exactly the point z_{NS}. In the algebra $G(z)$ we have the following generating operators.

$$\begin{aligned}
u^5 H_1(z) &= -t_{12} - ut_{13} - u^4 t_{14} - \frac{u^4 t_{15}}{u^2+1} - \frac{u^4 t_{16}}{u+1} \\
u^5 H_2(z) &= t_{12} + \frac{ut_{23}}{u-1} + \frac{u^4 t_{24}}{u^4-1} + \frac{u^4 t_{25}}{u^4-u^2-1} + \frac{u^4 t_{26}}{u^4-u^2-1} \\
u^4 H_3(z) &= t_{13} + \frac{t_{23}}{1-u} + \frac{u^3 t_{34}}{u^3-1} + \frac{u^3 t_{35}}{u^3-u^2-1} + \frac{u^3 t_{36}}{u^3-u^2-1} \\
u^3 H_4(z) &= u^2 t_{14} + \frac{u^2 t_{24}}{1-u^4} + \frac{u^2 t_{34}}{1-u^3} - t_{45} - ut_{46} \\
u^3 H_5(z) &= \frac{u^2 t_{15}}{u^2+1} + \frac{u^2 t_{25}}{u^2+1-u^4} + \frac{u^2 t_{35}}{u^2+1-u^3} + t_{45} + \frac{ut_{56}}{u-1} \\
u^2 H_6(z) &= \frac{ut_{16}}{u+1} + \frac{ut_{26}}{u+1-u^4} + \frac{ut_{36}}{u+1-u^3} + t_{46} + \frac{t_{56}}{1-u}
\end{aligned}$$

Taking the limit as $u \to 0$ we obtain 4 linearly independent operators,

$$\begin{aligned}
L_1 &= t_{12}, \\
L_2 &= t_{13} + t_{23} \\
L_4 &= t_{45}, \\
L_5 &= t_{46} + t_{56}.
\end{aligned}$$

We know by Proposition 2.4.2 that there are 5 linearly independent operators. By taking the limit of the operator $u(H_1(z) + H_2(z) + H_3(z))$, we obtain

$$L_3 = t_{14} + t_{15} + t_{16} + t_{24} + t_{25} + t_{26} + t_{34} + t_{35} + t_{36}.$$

These 5 operators generate the entire Gaudin subalgebra at the point z_{NS}. In Appendix A.1 we provide Magma code which computes the joint eigenspaces of the operators L_1, L_2, L_3, L_4, L_5. The output gives eight, 1-dimensional eigenspaces and four, 2-dimensional eigenspaces. The matrices representing the operators L_1, L_2, L_3, L_4, L_5, their eigenvalues and eigenvectors are also summarised in Appendix A, Table A.1.

Remark 2.4.10. This example is the smallest for which a Gaudin algebra fails to have simple spectrum on an irreducible S_n module. The reason for this is that the eigenvalues of the Gaudin subalgebras corresponding to points in $M_{n-1} \subset \overline{M}_{0,n+1}(\mathbb{C})$ (the most degenerate points) are related to the restriction multiplicities determined by the bracketing. In the present case, the bracketing $(((12)3)((45)6))$ tells us we should look at the restriction to the subgroup $S_3 \times S_3 \subset S_6$. The restriction of $S(3, 2, 1)$ in this case is not multiplicity free (it contains two copies of $S(2, 1) \otimes S(2, 1)$. This is the first time such a restriction to a Young subgroup has multiplicities greater than 1.

Remark 2.4.10 leads us to conjecture the following.

Conjecture 2.4.11. *To each full bracketing of $[n]$ we associate a tower of Young subgroups. The corresponding Gaudin subalgebra has image inside $\mathbb{C}S_n$ which generates the* Gelfand-Zetlin subalgebra *of this tower of subgroups.*

Recall that the Gelfand-Zetlin subalgebra of a tower of subgroups is the algebra generated by the centres. This conjecture and an investigation of the associated higher Hamiltonians will form future work of the author and is not considered further in this thesis.

Chapter 3

Bethe algebras and crystals

In this chapter we recall the definition of the Bethe algebras and crystals for $\mathfrak{gl}_r$. The Bethe algebras are commutative subalgebras of $\mathrm{End}(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$ containing the Gaudin Hamiltonians. In the same way as in Chapter 2 we can organise the spectrum of the Bethe algebras into a finite family over $M_{0,n+1}(\mathbb{C})$, ultimately it will be the monodromy of this family we relate to crystals. We will recall some combinatorial definitions and facts which we will need to formulate the main theorems of this thesis at the end of the chapter.

3.1 Bethe algebras

In Section 2.4.4 we saw the compactified Gaudin spectrum may be ramified over points in $\overline{M}_{0,n+1}(\mathbb{R})$ because the Gaudin Hamiltonians have simultaneous eigenspaces which jump dimension at certain points, in other words the algebra generated by the Gaudin Hamiltonians (and their limits) in $\mathrm{End}(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$ may not be maximally commutative. In order to resolve this issue, we consider slightly larger algebras, the Bethe algebras, whose definition we recall here. The Bethe algebras were constructed in [FFR94] and have been intensively studied by Mukhin, Tarasov and Varchenko and it is their exposition which we follow (see for example [MTV09a]).

3.1.1 Modules for the current algebra

Let $\mathfrak{gl}_r[t] = \mathfrak{gl}_r \otimes \mathbb{C}[t]$ be the *current algebra* of polynomials with coefficients in $\mathfrak{gl}_r$. For any complex number $w \in \mathbb{C}$ define the surjection $\rho_w : \mathfrak{gl}_r[t] \longrightarrow \mathfrak{gl}_r$, by evaluating a polynomial in t at w.

Definition 3.1.1. Let M be a $\mathfrak{gl}_r$ module and w a complex number. The *evaluation module* at w associated to M is the pullback of M along ρ_w. This $\mathfrak{gl}_r[t]$-module will be denoted $M(w)$.

The name comes from the fact that t acts as multiplication by w.

3.1.2 The universal Bethe algebra

In order to define the Bethe algebras, a universal version will be defined first, after which we will be able to specialise. This formulation is due to Talalaev [Tal06]. Let u be a formal variable and for any $x \in \mathfrak{gl}_r$ define the generating function

$$x(u;t) = x(u) = \sum_{s=0}^{\infty} (x \otimes t^s) u^{-s-1} = \frac{x}{u-t},$$

which is an element of the ring $U(\mathfrak{gl}_r[t])[[u^{-1}]]$, of formal power series in u^{-1}. Let $A = (a_{ij})$ be an $n \times n$-matrix with entries in a noncommutative ring, for the purposes of this thesis, we define the determinant of A to be

$$\det A = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

If ∂ is differentiation with respect to u then we can define the following differential operator on $U(\mathfrak{gl}_r[t])[[u^{-1}]]$,

$$\mathcal{D}(u,\partial;t) = \det \begin{pmatrix} \partial - e_{11}(u) & -e_{21}(u) & \cdots & -e_{r1}(u) \\ -e_{12}(u) & \partial - e_{22}(u) & \cdots & -e_{r2}(u) \\ \vdots & \vdots & \ddots & \vdots \\ -e_{1r}(u) & -e_{2r}(u) & \cdots & \partial - e_{rr}(u) \end{pmatrix}$$

where e_{ij} are the standard generators for $\mathfrak{gl}_r$. The operator $\mathcal{D}(u,\partial;t)$ has the form

$$\mathcal{D}(u,\partial;t) = \partial^r + \sum_{i=1}^{r} B_i(u) \partial^{r-i},$$

for some power series with coefficients $B_{is} \in \mathfrak{gl}_r[t]$,

$$B_i(u;t) = B_i(u) = \sum_{s=0}^{\infty} B_{is} u^{-s-1}.$$

Example 3.1.2. By definition the noncommutative determinant $\mathcal{D}(u,\partial;t)$ is given by

$$\mathcal{D}(u,\partial;t) = \sum_{\sigma \in S_r} (-1)^{\sigma} \prod_{i=1}^{r} \left(\delta_{i\sigma(i)} \partial - e_{\sigma(i)i}(u) \right). \tag{3.1.1}$$

If we wish to calculate $B_1(u)$ it is enough to look at the term when $\sigma = \mathrm{id}$. This is

$$\left(\partial - \frac{e_{11}}{u-t} \right) \left(\partial - \frac{e_{22}}{u-t} \right) \cdots \left(\partial - \frac{e_{rr}}{u-t} \right). \tag{3.1.2}$$

Since we would like to calculate the coefficient of ∂^{r-1}, we concentrate on the terms with $r-1$ factors of ∂,

$$-\sum_{i=1}^{r} \partial^{i-1} \frac{e_{ii}}{u-t} \partial^{r-i} = -\sum_{i=1}^{r} \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} \frac{k! e_{ii}}{(u-t)^{k+1}} \partial^{r-k-1}. \tag{3.1.3}$$

We can easily see the term of degree $r-1$ in ∂ is

$$B_1(u) = -\sum_{i=1}^{r} e_{ii}(u) = -\frac{\sum_i e_{ii}}{u-t}.$$

Example 3.1.3. We can now calculate $B_2(u)$, which is the coefficient of ∂^{r-2}. From (3.1.1) we see that we need to look at the terms corresponding to the cases when $\sigma = \mathrm{id}$ and when σ is a transposition. First, when $\sigma = \mathrm{id}$, the terms in expression (3.1.3) give the following contribution of coefficients of ∂^{r-2} (i.e when $k = 1$),

$$\sum_{i=2}^{r} (i-1) \frac{e_{ii}}{(u-t)^2}. \tag{3.1.4}$$

Now looking back at (3.1.2), we also need to look at the terms which have $r-2$ factors of ∂. This is

$$\sum_{i<j} \partial^{i-1} \frac{e_{ii}}{u-t} \partial^{j-i-1} \frac{e_{jj}}{u-t} \partial^{r-j}.$$

Upon commuting the ∂ to the right, we end up with an expression which has as the coefficient of ∂^{r-2}

$$\sum_{i<j} \frac{e_{ii} e_{jj}}{(u-t)^2}. \tag{3.1.5}$$

Now we need to look at the terms of (3.1.1) where σ is a transposition. These are

$$-\sum_{i>j} \left(\partial - \frac{e_{11}}{u-t}\right) \left(\partial - \frac{e_{22}}{u-t}\right) \cdots \frac{e_{ij}}{u-t} \cdots \frac{e_{ji}}{u-t} \cdots \left(\partial - \frac{e_{rr}}{u-t}\right).$$

We are only interested in the terms with at least $r-2$ factors ∂, so we are looking at

$$-\sum_{i>j} \partial^{j-1} \frac{e_{ij}}{u-t} \partial^{i-j-1} \frac{e_{ji}}{u-t} \partial^{r-j}.$$

Upon rearranging we find the coefficient of ∂^{r-2} to be

$$-\sum_{i>j} \frac{e_{ij} e_{ji}}{(u-t)^2}. \tag{3.1.6}$$

Summing up the contributions coming from (3.1.4), (3.1.5) and (3.1.6) we obtain

$$\begin{aligned} B_2(u) &= \frac{\sum_{i=1}^{r} \left((i-1) e_{ii} + \sum_{j>i} e_{ii} e_{jj} - \sum_{j<i} e_{ij} e_{ji} \right)}{(u-t)^2} \\ &= \frac{\sum_{1 \le i < j \le r} (e_{ii} + e_{ii} e_{jj} - e_{ij} e_{ji})}{(u-t)^2}. \end{aligned}$$

Definition 3.1.4. The *universal Bethe algebra* is the subalgebra A, of $U(\mathfrak{gl}_r[t])$ generated by the coefficients B_{is} where $1 \leq i \leq r$ and $s \geq 0$. For an A-module M, we call the image of A in $\mathrm{End}(M)$ the *Bethe algebra associated to M*.

3.1.3 Specialisations of the Bethe algebra

Let Δ be the coproduct for $U(\mathfrak{gl}_r[t])$. For $n \geq 2$ define

$$\Delta^{(n)} : U(\mathfrak{gl}_r[t]) \longrightarrow U(\mathfrak{gl}_r[t])^{\otimes n}$$

inductively by $\Delta^{(2)} = \Delta$ and $\Delta^{(n+1)} = (\Delta \otimes \mathrm{id}^{\otimes n-1}) \circ \Delta^{(n)}$. Thus for $x \in \mathfrak{gl}_r[t]$, $\Delta^{(n)}(x) = \sum_{a=1}^{n} x^{(a)}$. Fix $z = (z_1, z_2, \ldots, z_n) \in X_n$ an n-tuple of distinct complex numbers.

Definition 3.1.5. The *Bethe algebra at z* is the image of the universal Bethe algebra A under the surjection

$$\rho_z = \rho_{z_1} \otimes \rho_{z_2} \otimes \cdots \otimes \rho_{z_n} \circ \Delta^{(n)} : U(\mathfrak{gl}_r[t]) \longrightarrow U(\mathfrak{gl}_r)^{\otimes n}.$$

We denote the image by $A(z)$.

Suppose $M_1, M_2, \ldots, M_n$ are $\mathfrak{gl}_r[t]$-modules. By definition, the action of A on $M_1(z_1) \otimes M_2(z_2) \otimes \cdots M_n(z_n)$ factors through $A(z)$.

Proposition 3.1.6 ([MTV06, Appendix B])**.** *The image of the Gaudin algebra $G(z)$ in $U(\mathfrak{gl}_r)^{\otimes n}$ is contained in $A(z)$.*

Lemma 3.1.7. *The Bethe algebras $A(z)$ are invariant under the action of the group* Aff_1*: if $\alpha \in \mathbb{C}^\times, \beta \in \mathbb{C}$, then $A(\alpha z + \beta) = A(z)$.*

Proof. We first make some general observations. For a $\mathbb{C}$-algebra R, consider the ring $R[[u^{-1}]][\partial]$ of differential operators with coefficients in formal power series of u^{-1}. Suppose we have an element $P(u, \partial) \in R[[u^{-1}]][\partial]$, and linear elements $a\partial + b$ and $cu + d$, for $a, c \in \mathbb{C}^\times$ and $b, d \in \mathbb{C}$. In order for the expression $P(cu + d, a\partial + b)$ to be well defined we need that they satisfy the relation $[a\partial + b, cu + d] = 1$, i.e. we need $a = c^{-1}$. More precisely, the map

$$\begin{aligned} R[[u^{-1}]][\partial] &\longrightarrow R[[u^{-1}]][\partial] \\ u^{-1} &\longmapsto (cu + d)^{-1} \\ \partial &\longmapsto a\partial + b \end{aligned}$$

is a homomorphism of R-algebras whenever $a = c^{-1}$.

First we will prove $A(z + \beta) = A(z)$. Observe if $x \in \mathfrak{gl}_r[t]$ then

$$x(u - \beta; t - \beta) = \frac{x}{u - \beta - (t - \beta)} = x(u; t),$$

so $\mathcal{D}(u+\beta, \partial; t-\beta) = \mathcal{D}(u, \partial; t)$. Here we have noted that $[\partial, u+\beta] = 1$. Equating coefficients we see that $B_i(u+\beta; t-\beta) = B_i(u;t)$. So

$$\rho_{z+\beta} B_i(u;t) = \rho_{z+\beta} B_i(u+\beta; t-\beta) = \rho_z B_i(u+\beta; t).$$

Thus $\rho_{z+\beta}(B_{is})$ is the coefficient of u^{-s-1} in $\rho_z B_i(u+\beta;t)$ and $\rho_z(B_{is})$ is the coefficient of $(u+\beta)^{-s-1}$. Expanding this out we obtain

$$\rho_{z+\beta}(B_{is}) = \sum_{k=0}^{s} \binom{s}{k} \beta^{s-k} \rho_z(B_{ik}).$$

The fact that $A(\alpha z) = A(z)$ is similar. First note that we can relate the two determinants $\mathcal{D}(\alpha^{-1}u, \alpha\partial; \alpha^{-1}t) = \alpha^r \mathcal{D}(u, \partial; t)$, again noting the commutator $[\alpha\partial, \alpha^{-1}u] = 1$. Thus we see that $\alpha^i B_i(u;t) = B_i(\alpha^{-1}u; \alpha^{-1}t)$, hence

$$\rho_{\alpha z} B_i(u;t) = \alpha^{-i} \rho_{\alpha z} B_i(\alpha^{-1}u; \alpha^{-1}t) = \alpha^{-i} \rho_z B_i(\alpha^{-1}u; t).$$

So we see $\rho_{\alpha z}(B_{is}) = \alpha^{s-i+1} \rho_z(B_{is})$. □

There are two important properties of the universal Bethe algebra which we summarise in the following proposition.

Proposition 3.1.8 ([MTV06, Propositions 8.2 and 8.3]). *The universal Bethe algebra A is a commutative subalgebra of $U(\mathfrak{gl}_r[t])$. Furthermore it lies in the centraliser of the subalgebra $U(\mathfrak{gl}_r)$.*

As a result, for any $\mathfrak{gl}_r[t]$-module M, and any weight λ, the weight spaces M_λ, M^{sing} and $M_\lambda^{\mathsf{sing}} \subset M$ are A-submodules. Let $\lambda_\bullet$ be a sequence of partitions with at most r rows. As a special case of Definition 3.1.4, for $z = (z_1, z_2, \ldots, z_n) \in X_n$ we denote the Bethe algebra associated to

$$L(\lambda_\bullet; z)_\mu^{\mathsf{sing}} = [L(\lambda_1)(z_1) \otimes L(\lambda_2)(z_2) \otimes \cdots \otimes L(\lambda_n)(z_n)]_\mu^{\mathrm{sing}}$$

by $A(\lambda_\bullet; z)_\mu$. Note that as a $U(\mathfrak{gl}_r)^{\otimes n}$-representation $L(\lambda_\bullet; z)_\mu^{\mathsf{sing}}$ is the same as $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$. These are our main objects of study.

3.1.4 The Bethe spectrum

Fix a sequence of n partitions $\lambda_\bullet$, and an auxiliary partition μ (each with at most r rows). Let $D = \prod_{a \neq b}(x_a - x_b) \in \mathbb{C}[x_1, \ldots, x_n]$. The ring of functions on X_n is $\mathbb{C}[x_1, \ldots, x_n, D^{-1}]$. The image of A under the morphism

$$\rho = \rho_{x_1} \otimes \rho_{x_2} \otimes \cdots \otimes \rho_{x_n} \circ \Delta^n : U(\mathfrak{gl}_r[t]) \longrightarrow U(\mathfrak{gl}_r)^{\otimes n} \otimes \mathbb{C}[X_n]$$

is denoted $A(X_n)$ and has $A(z)$ as its fibre over $z \in X_n$. As demonstrated, the action of Aff_1 on X_n extends to an action of $A(X_n)$ which identifies the fibres over the orbits.

Taking the invariants of Aff_1 results in a sheaf A_n over $M_{0,n+1}(\mathbb{C})$. As in Section 2.2.4 we can consider the image $A(\lambda_\bullet)_\mu$ of A_n in $\mathcal{E}nd(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$, the trivial bundle over $M_{0,n+1}(\mathbb{C})$ with fibre $\mathrm{End}(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$. This makes $A(\lambda_\bullet)_\mu$ a $\mathbb{C}[M_{0,n+1}(\mathbb{C})]$-algebra.

Definition 3.1.9. The spectrum of $A(\lambda_\bullet)_\mu$ and is denoted $\mathcal{A}(\lambda_\bullet)_\mu$. Since $A(\lambda_\bullet)_\mu$ is an algebra over $M_{0,n+1}(\mathbb{C})$ is comes with a structure morphism, the *Bethe spectrum* denoted $\pi_{\lambda_\bullet,\mu} : \mathcal{A}(\lambda_\bullet)_\mu \longrightarrow M_{0,n+1}(\mathbb{C})$.

The fibre over $z \in M_{0,n+1}(\mathbb{C})$ of the Bethe spectrum is the spectrum of the commutative algebra $A(\lambda_\bullet; z)_\mu$ which we denote $\mathcal{A}(\lambda_\bullet; z)_\mu$. The key result about Bethe algebras we will use is the following set of facts by Mukhin, Tarasov and Varchenko.

Theorem 3.1.10 ([MTV09a], Corollary 6.3)**.** *Let $z = (z_1, z_2, \ldots, z_n)$ be an n-tuple of distinct complex numbers*

(i) The dimension of the Bethe algebra $A(\lambda_\bullet; z)_\mu$ is $c^\mu_{\lambda_\bullet}$, in particular it does not depend on z.

(ii) If $z \in M_{0,n+1}(\mathbb{R})$, the Bethe algebra $A(\lambda_\bullet; z)_\mu$ has simple spectrum.

The integers $c^\mu_{\lambda_\bullet}$ are the *Littlewood-Richardson coefficients*, see Section 4.1.3 for a definition. For us, the most important consequence of this theorem is the following.

Corollary 3.1.11. *The Bethe spectrum $\pi_{\lambda_\bullet,\mu} : \mathcal{A}(\lambda_\bullet)_\mu \longrightarrow M_{0,n+1}(\mathbb{C})$ is a finite and flat map. Upon restriction to the real points $M_{0,n+1}(\mathbb{R})$, the Bethe spectrum is an unbranched topological covering space of degree $c^\mu_{\lambda_\bullet}$.*

Proof. The algebra $A(\lambda_\bullet)_\mu$ is defined as a $\mathbb{C}[M_{0,n+1}(\mathbb{C})]$-submodule of the free module $\mathcal{E}nd(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$ which is finitely generated. Since $\mathbb{C}[M_{0,n+1}(\mathbb{C})]$ is Noetherian, $A(\lambda_\bullet)_\mu$ is finitely generated. This demonstrates $\pi_{\lambda_\bullet,\mu}$ is a finite map. In order to show flatness we use Theorem 3.1.10 (i), which says the dimension of the fibres of $\pi_{\lambda_\bullet,\mu}$ is constant at closed points, and apply Proposition C.1.5 which says $\pi_{\lambda_\bullet,\mu}$ is hence flat. The fact that the restriction to real points is a topological covering space follows from Theorem 3.1.10 (ii) which implies that $\mathcal{A}(\lambda_\bullet; z)_\mu$ is a reduced set of points. $\square$

We now consider the special case when $\lambda_\bullet = (\square^n)$. In this case the Bethe algebra $A(z)_\mu$ acts on the module $S(\mu) = [V^{\otimes n}]_\mu^{\mathsf{sing}}$.

Theorem 3.1.12 ([MTV10, Theorem 3.2])**.** *Let $z = (z_1, z_2, \ldots, z_n)$ be an n-tuple of distinct complex numbers. The Gaudin Hamiltonians generate the Bethe algebra $A(z)_\mu$.*

Corollary 3.1.13. *Let $z = (z_1, z_2, \ldots, z_n)$ be an n-tuple of distinct real numbers, then the Gaudin Hamiltonians have simple spectrum on $S(\mu)$ for any partition μ of n.*

Proof. By Theorem 3.1.12, $A(z)_\mu$ equals the algebra generated by the image of $G(z)$ in $\mathrm{End}(S(\mu))$ and by Theorem 3.1.10 (ii) this has simple spectrum. $\square$

3.2 Tableaux

To discuss crystals, cactus actions, and to formulate the main results in this thesis requires various combinatorial definitions and facts about tableaux, which we briefly recall here.

3.2.1 Fillings and tableaux

Let $\mathcal{A}$ be a totally ordered alphabet. Most of the time we will take as our alphabet the positive integers $[n]$ and this is always the alphabet we will use in examples. A *filling* of a skew-shape $\lambda\backslash\mu$ is an assignment of letters from $\mathcal{A}$ to the boxes of $\lambda\backslash\mu$. Three examples of fillings for $(4,3,3,1)\backslash(3,1,1)$ are

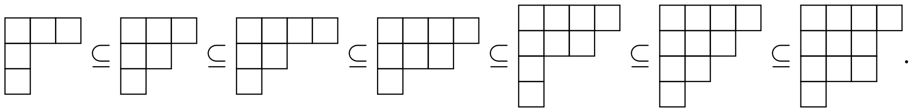
$$R = \begin{array}{cccc} & & & \boxed{3} \\ & \boxed{2} & \boxed{2} & \\ & \boxed{3} & \boxed{4} & \\ \boxed{1} & & & \end{array}, \quad S = \begin{array}{cccc} & & & \boxed{2} \\ & \boxed{1} & \boxed{3} & \\ & \boxed{5} & \boxed{6} & \\ \boxed{4} & & & \end{array} \quad \text{and} \quad T = \begin{array}{cccc} & & & \boxed{3} \\ & \boxed{5} & \boxed{2} & \\ & \boxed{5} & \boxed{4} & \\ \boxed{1} & & & \end{array}. \tag{3.2.1}$$

Definition 3.2.1. A *semistandard tableau* (which we will often simply call a *tableau*) of shape $\lambda\backslash\mu$ is a filling such that rows are weakly increasing and columns are strictly increasing with respect to the total order on $\mathcal{A}$. A *standard tableau* of shape $\lambda\backslash\mu$ is a filling using the numbers 1 to $|\lambda\backslash\mu|$ such that rows and columns are increasing and such that each number appears only once. In the examples (3.2.1), R is semistandard, S is standard (and semistandard) and T is neither.

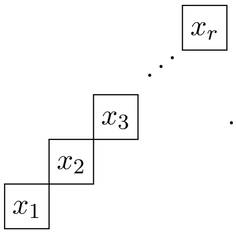
We will denote the set of standard $\lambda\backslash\mu$-tableaux by $\mathtt{SYT}(\lambda\backslash\mu)$ and the semistandard $\lambda\backslash\mu$-tableaux by $\mathtt{SSYT}(\lambda\backslash\mu)$.

Remark 3.2.2. Something we will often do in the combinatorial algorithms described below is compare the relative size of the entries in two given boxes of a tableau. For standard tableaux there is no ambiguity as no two boxes contain the same element. However for semistandard tableaux if two boxes contain the same element of $\mathcal{A}$, we will say that the entry of the leftmost box is *smaller*. Since columns are strictly increasing we always have a leftmost box. This seems like an arbitrary choice however we will quickly see that this is in fact the only sensible choice.

Remark 3.2.3. An important observation that we will use many times is the fact that a standard tableau of shape $\lambda\backslash\mu$ is the same thing as a sequence of partitions $\nu_1 \subseteq \nu_2 \subseteq \ldots \subseteq \nu_r$ where $\nu_{i+1}\backslash\nu_i$ is a single box and $\nu_1 = \mu$ and $\nu_r = \lambda$. We call such a sequence a *growth sequence*. As an example, take S as in (3.2.1), then we identify this standard tableau with the growth sequence

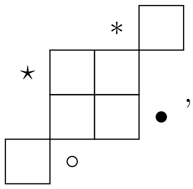


Remark 3.2.4. Words in the alphabet $\mathcal{A}$ are identified with semistandard tableaux of *diagonal* shape. That is, the word $x_1 x_2 \cdots x_r$ is identified with the semistandard tableau



Definition 3.2.5. Given a skew-shape $\lambda \backslash \mu$ we define an *addable node* to be a box, when added to the diagram, form another skew-shape. An addable node occurring on the south-eastern edge of the diagram is an addable *outside* node and one occurring on the north-western edge an addable *inside* node.

For example in the diagram



(3.2.2)

the symbols $*$ and $\circ$ describe addable inside and outside nodes respectively and $\star$ and $\bullet$ describe non-addable nodes.

3.2.2 Slide equivalence

A *jeu de taquin slide* or simply a *slide* is a transformation of one semistandard tableaux into another due to Schützenberger [Sch77]. These slides come in two variations. First a *forward slide* is performed by choosing an addable outside node (see Remark 3.2.5) and then sliding the box with the largest entry of the boxes above and to the left into the vacant spot. The largest entry is well defined by Remark 3.2.2. This sliding process is now repeated with the now vacated box until the vacated box becomes an addable inside node.

As an example take R from (3.2.1) and make a forward slide into the addable outside node $\circ$ defined in (3.2.2). By performing the sliding procedure we obtain the following sequence of tableau.

$$\begin{array}{cccc} & & & 3 \\ & 2 & 2 & \\ & 3 & 4 & \\ 1 & \circ & & \end{array} \rightsquigarrow \begin{array}{cccc} & & & 3 \\ & 2 & 2 & \\ & \circ & 4 & \\ 1 & 3 & & \end{array} \rightsquigarrow \begin{array}{cccc} & & & 3 \\ & \circ & 2 & \\ & 2 & 4 & \\ 1 & 3 & & \end{array} .$$

A *reverse slide* is the same except now one starts with an addable inside corner of the skew shape and slide the smallest of the two boxes below and to the right into the vacant position. This reverse sliding process is now repeated with the now vacated box until the vacated box becomes an addable outside node.

Definition 3.2.6. Two semistandard tableaux, T and T', are called *slide equivalent*, denoted $T \sim_S T'$, if one can be transformed into the other by a sequence of jeu de taquin slides.

The following theorem is often referred to as *the first fundamental theorem of jeu de taquin*. For a proof see Claim 2 in Section 1.2 of [Ful97] or the remarks after Theorem 2.13 in [Hai92].

Theorem 3.2.7. *Every slide equivalence class contains exactly one tableau of normal shape.*

Given a tableau T, we call the unique tableau of normal shape, slide equivalent to T, the *rectification* of T and denote it $\texttt{Rect}(T)$. For example, the rectification of

$$R = \begin{array}{cccc} & & & 3 \\ & 2 & 2 & \\ & 3 & 4 & \\ 1 & & & \end{array} \quad \text{is} \quad \texttt{Rect}(R) = \begin{array}{cccc} 1 & 2 & 2 & 3 \\ 3 & 4 & & \end{array}.$$

We can also characterise exactly when two words (considered as tableaux as in Remark 3.2.4) are slide equivalent.

Proposition 3.2.8. *Two words are slide equivalent if and only if they differ by a sequence of* Knuth moves *which are local operations allowing the following swaps,*

$$\begin{aligned} xyz &\mapsto xzy \quad \text{if } z < x \leq y \text{ or } y < x \leq z, \\ xyz &\mapsto yxz \quad \text{if } x \leq z < y \text{ or } y \leq z < x. \end{aligned}$$

Proof. See [Ful97, Section 2.1]. $\square$

3.2.3 The Schützenberger involution

We now describe an involution on semistandard tableaux based on the jeu de taquin slides described in Section 3.2.2. The definition is due to Schützenberger [Sch77]. Denote by $a \mapsto a^*$ the unique involution on $\mathcal{A}$ that reverses the total order (i.e. if $\mathcal{A} = [n]$ then the involution is given by $i^* = n + 1 - i$).

Let T be of shape $\lambda \backslash \mu$. If T consists of a single box, let a denote its entry and let $\xi(T)$ be the tableau of same shape with entry a^*. If T has more than one box we construct a tableau $\xi(T)$ of the same shape, inductively on the number of boxes as follows:

1. Start with $\xi(T)$ as the shape $\lambda \backslash \mu$ without any filling.
2. Remove from T the box with smallest entry a.
3. Denote the newly vacated addable inside node by $\circ$.
4. Reverse slide $T \backslash \{\circ\}$ into $\circ$. We obtain a new tableau T' of shape $\lambda' \backslash \mu$.
5. The shape $\lambda \backslash \lambda'$ determines the box in $\lambda \backslash \mu$ vacated by the reverse slide.
6. Fill the corresponding box in $\xi(T)$ with a^*.
7. The remaining unfilled boxes in $\xi(T)$ are of shape $\lambda' \backslash \mu$. Fill with $\xi(T')$.

Recall that the smallest entry is well defined by Remark 3.2.2 and by the same Remark also defines an addable inside node once removed.

Definition 3.2.9. The map ξ sending $T \in \mathtt{SSYT}(\lambda \backslash \mu)$ to $\xi(T) \in \mathtt{SSYT}(\lambda \backslash \mu)$ is called the *Schützenberger involution*.

A priori it is not clear that ξ deserves to be called an involution however this turns out to be the case (see Proposition 3.2.10) and will be made obvious once we describe growth diagrams.

As an example we will calculate the Schützenberger involution of R as in (3.2.1). We will take as our alphabet $\mathcal{A} = [9]$. We begin by emptying the box with a 1, making a reverse slide, then entering $1^* = 9$ into the appropriate box. Since the box with 1 is both an addable inside and outside node the slide is trivial.

$$R = \begin{ytableau} \none & \none & \none & 3 \\ \none & 2 & 2 \\ \none & 3 & 4 \\ 1 \end{ytableau} \quad \rightsquigarrow \quad \begin{ytableau} \none & \none & \none & 3 \\ \none & 2 & 2 \\ \none & 3 & 4 \\ \none[\circ] \end{ytableau} \qquad \xi(R) = \begin{ytableau} \none & \none & \none & \\ \none & & \\ \none & & \\ 9 \end{ytableau}$$

Now we repeat the process with the box containing the leftmost 2 in the new tableau and insert a $2^* = 8$ into $\xi(R)$.

$$R' = \begin{array}{ccc} & & 3 \\ 2 & 2 & \\ 3 & 4 & \end{array} \rightsquigarrow \begin{array}{ccc} & & 3 \\ \circ & 2 & \\ 3 & 4 & \end{array}$$

$$\rightsquigarrow \begin{array}{ccc} & & 3 \\ 2 & \circ & \\ 3 & 4 & \end{array} \rightsquigarrow \begin{array}{ccc} & & 3 \\ 2 & 4 & \\ 3 & \circ & \end{array} \qquad \xi(R) = \begin{array}{cccc} & & & \square \\ & \square & \square & \\ & \square & 8 & \\ 9 & & & \end{array}$$

Again, we remove the next two and repeat with the new tableau.

$$R'' = \begin{array}{ccc} & & 3 \\ 2 & 4 & \\ 3 & & \end{array} \rightsquigarrow \begin{array}{ccc} & & 3 \\ \circ & 4 & \\ 3 & & \end{array}$$

$$\rightsquigarrow \begin{array}{ccc} & & 3 \\ 3 & 4 & \\ \circ & & \end{array} \qquad \xi(R) = \begin{array}{cccc} & & & \square \\ & \square & \square & \\ & 8 & 8 & \\ 9 & & & \end{array}$$

The next entry we take is the leftmost 3, note that $3^* = 7$.

$$R''' = \begin{array}{ccc} & & 3 \\ 3 & 4 & \end{array} \rightsquigarrow \begin{array}{ccc} & & 3 \\ \circ & 4 & \end{array}$$

$$\rightsquigarrow \begin{array}{ccc} & & 3 \\ 4 & \circ & \end{array} \qquad \xi(R) = \begin{array}{cccc} & & & \square \\ & \square & 7 & \\ & 8 & 8 & \\ 9 & & & \end{array}$$

Since all remaining boxes are both addable inside and outside nodes when removed, we simply fill the corresponding boxes in $\xi(R)$ with $3^* = 7$ and $4^* = 6$ to obtain

$$\xi(R) = \begin{array}{cccc} & & & 7 \\ & 6 & 7 & \\ & 8 & 8 & \\ 9 & & & \end{array}.$$

One can check that $\xi(\xi(R)) = R$.

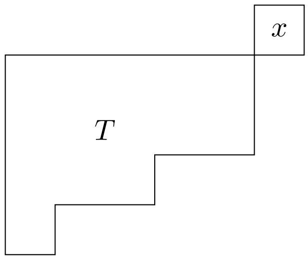


Figure 3.1: The skew-tableau $T * x$

Proposition 3.2.10 ([Sch77, p. 8.2]). *The operation ξ is a shape preserving involution of semistandard Young tableaux.*

Proof. We give a proof in Section 4.3.3 once we have defined growth diagrams. $\square$

Remark 3.2.11. The Schützenberger involution will appear a number of times in later sections, however it will play differing roles depending on whether we consider it as a involution of semistandard tableaux or standard tableaux. To make this distinction more obvious we will use the notation ξ only for semistandard tableaux and the notation `evac` when applying the involution to standard tableaux.

3.2.4 Insertion

In this subsection we recall the *insertion* of a letter x into a tableau T of normal shape. This process defines a new tableau $T \leftarrow x$ as follows: create the skew-tableau $T * x$ as depicted in Figure 3.1 and define the tableau $T \leftarrow x$ to be $\texttt{Rect}(T * x)$.

Remark 3.2.12. More generally, given two tableaux, S and T, of normal shape, we can form the tableau $S * T$ and define $S \cdot T = \texttt{Rect}(S * T)$. This defines an associative product on the set of semistandard tableau of normal shape which is usually referred to as the *plactic monoid*. See [Ful97] for details.

As an example let T be the tableau

$$\begin{array}{|c|c|c|}\hline 2 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array},$$

and let $x = 3$. We obtain

$$T * 3 = \begin{array}{cccc}\cline{4-4} & & & \multicolumn{1}{|c|}{3} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \\ \cline{1-3} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} & & \\ \cline{1-2}\end{array} \quad \rightsquigarrow \quad \begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} = \texttt{Rect}(T * 3) = T \leftarrow 3.$$

Remark 3.2.13. There is a second, equivalent, way to define the insertion tableau $T \leftarrow x$. First, if x can be placed at the end of the first row we do so and output this as the new tableau. Otherwise we place x into the leftmost box in the first row with an entry $y > x$. We now repeat this procedure with y and the second row. We continue this way, inserting and bumping out entries until we are able to simply place the previously bumped out entry onto the end of a (possibly empty) row.

3.2.5 The RSK algorithm

The *RSK algorithm* gives a bijection between words of length n in $\mathcal{A}$ and pairs of a semistandard and standard tableaux of the same normal shape with n boxes. The algorithm was originally defined for permutations by Robinson [Rob38] and earlier by Schensted [Sch61], and it was later generalised by Knuth [Knu70]. The definition uses the insertion algorithm.

Definition 3.2.14. Given a word $w = x_1 x_2 \cdots x_n$ we define its *P-symbol* or *insertion tableau* to be the semistandard tableau defined by the successive insertion

$$\mathtt{P}(w) = (\dots((\emptyset \leftarrow x_1) \leftarrow x_2) \dots \leftarrow x_n),$$

and its *Q-symbol* or *recording tableau* to be the standard tableau, $\mathtt{Q}(w)$, given by the growth sequence

$$\emptyset \subset \mathrm{sh}(P_1) \subset \mathrm{sh}(P_2) \subset \cdots \subset \mathrm{sh}(P_n),$$

where $P_i = \mathtt{P}(x_1 x_2 \cdots x_i)$. The name recording tableau comes from the fact it records in which order each box of $P(w)$ was added during the insertion process. By construction $\mathtt{P}(w)$ is a semistandard tableaux with entries in $\mathcal{A}$ and $\mathtt{Q}(w)$ is a standard tableaux of the same shape.

Remark 3.2.15. Recall that we can identify a word with a semistandard tableaux of diagonal shape. With this description in mind, $\mathtt{P}(w) = \mathtt{Rect}(w)$ and $P_i = \mathtt{Rect}(x_1 x_2 \cdots x_i)$. This description gives the following classification of the slide equivalence classes of words.

Proposition 3.2.16. *Two words considered as tableaux are slide equivalent if and only if their P-symbols agree.*

Proof. By Theorem 3.2.7 the slide equivalence class of a word, w, is determined by the unique tableau of normal shape slide equivalent to it. By definition this is $\mathtt{Rect}(w) = \mathtt{P}(w)$. Hence the words w and w' are slide equivalent if and only if $\mathtt{P}(w) = \mathtt{P}(w')$. □

Definition 3.2.17. The *Robinson-Schensted-Knuth (RSK) correspondence* is the assignment $\mathtt{RSK}(w) = (\mathtt{P}(w), \mathtt{Q}(w))$. Let $\mathtt{words}(n)$ denote the set of words of length n in the alphabet $\mathcal{A}$.

Theorem 3.2.18 ([Knu70, Theorem 2]). *The function*

$$\mathtt{RSK}\colon \mathtt{words}(n) \longrightarrow \bigsqcup_{|\lambda|=n} \mathtt{SSYT}(\lambda) \times \mathtt{SYT}(\lambda)$$

is a bijection.

Proof. See Section 4.1 of [Ful97]. □

Example 3.2.19. Suppose $w = 14321321$. We have the following sequence of pairs of tableaux defining $\mathtt{P}(w)$ and $\mathtt{Q}(w)$.

$$\begin{ytableau} 1 \end{ytableau}\ ,\ \begin{ytableau} 1 \end{ytableau}$$

$$\begin{ytableau} 1 & 4 \end{ytableau}\ ,\ \begin{ytableau} 1 & 2 \end{ytableau}$$

$$\begin{ytableau} 1 & 3 \\ 4 \end{ytableau}\ ,\ \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$

$$\begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}\ ,\ \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}$$

$$\begin{ytableau} 1 & 1 \\ 2 \\ 3 \\ 4 \end{ytableau}\ ,\ \begin{ytableau} 1 & 2 \\ 3 \\ 4 \\ 5 \end{ytableau}$$

$$\begin{ytableau} 1 & 1 & 3 \\ 2 \\ 3 \\ 4 \end{ytableau}\ ,\ \begin{ytableau} 1 & 2 & 6 \\ 3 \\ 4 \\ 5 \end{ytableau}$$

$$\begin{ytableau} 1 & 1 & 2 \\ 2 & 3 \\ 3 \\ 4 \end{ytableau}\ ,\ \begin{ytableau} 1 & 2 & 6 \\ 3 & 7 \\ 4 \\ 5 \end{ytableau}$$

$$\mathtt{P}(w) = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \cline{1-2} 3 & 3 & \\ \cline{1-2} 4 & & \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 7 & \\ \cline{1-2} 4 & 8 & \\ \cline{1-2} 5 & & \\ \cline{1-1}\end{array} = \mathtt{Q}(w)$$

3.2.6 Schützenberger involution and RSK

If $w = x_1 x_2 \dots x_n$ with $x_i \in \mathcal{A}$ then denote

$$w^* = x_n^* x_{n-1}^* \cdots x_1^*.$$

With this notation we have an amazing duality theorem.

Theorem 3.2.20. *If* $\mathtt{RSK}(w) = (P, Q)$ *then* $\mathtt{RSK}(w^*) = (\xi P, \mathtt{evac} Q)$.

Proof. See Section 1 of Appendix A in [Ful97] for the proof. □

We also have the following proposition that gives information about the Q-symbol of subwords. Denote by $T|_{r,s}$ the tableaux obtained from T by throwing away boxes filled with letters outside the range $[r, s]$.

Proposition 3.2.21. *Let* $w = x_1 x_2 \cdots x_n$ *be a word with* Q*-symbol* Q *and suppose that* $u = x_r x_{r+1} \cdots x_s$ *is a (contiguous) subword, then the* Q *symbol of* u *is* $\mathtt{Rect}(Q|_{r,s})$.

Proof. See Proposition 1 in Section 5.1 of [Ful97]. □

3.2.7 Representations of the symmetric group

In this section we recall some facts about the representation theory of the symmetric group. This approach to the representation theory of the symmetric group is originally due to Jucys [Juc74] and was developed by Okounkov and Vershik [OV96]. For proofs and a full exposition, see for example [Kle05, Chapter 2]. As in Example 2.2.13, let $S(\mu) = [V^{\otimes n}]_\mu^{\mathsf{sing}}$ which is the irreducible representation of S_n corresponding to the partition μ of n. Let $L_a = \sum_{b<a} (b, a)$ be the Jucys-Murphy elements. Let $S \in \mathtt{SYT}(\mu)$, then define $\mathrm{row}_S(a)$ and $\mathrm{col}_S(a)$ to be the row and column numbers of the box containing a in S. Define $c_S(a) = \mathrm{row}_S(a) - \mathrm{col}_S(a)$ to be the *content* of the box of S containing a.

Theorem 3.2.22. *The representation* $S(\mu)$ *and the operators* L_a *have the following properties.*

(i) The dimension of $S(\mu)$ *is* $d_\mu = \#\mathtt{SYT}(\mu)$.

(ii) The Jucys-Murphy operators are diagonalisable on S_μ *with one dimensional simultaneous eigenspaces.*

(iii) *The simultaneous eigenspaces can be canonically labelled by* $\mathtt{SYT}(\mu)$ *so that if* v_S *is an eigenvector labelled by* $S \in \mathtt{SYT}(\mu)$ *then,*

$$L_a v_S = c_S(a) v_S.$$

Define $z_{JM} \in \overline{M}_{0,n+1}(\mathbb{C})$ to be the point labelled by $(((12)3)\cdots n)$. Recall from Proposition 2.4.7 that the image of the Gaudin subalgebra $G(z_{JM})$ in $\mathrm{End}(S(\mu))$ coincides with the algebra generated by the Jucys-Murphy operators. Let $\Theta^\circ_{\mathrm{id}} \subset M_{0,n+1}(\mathbb{R})$ be the subset represented by curves with marked points at $z_1, z_2, \ldots, z_n$ and ∞ such that z_i is real and $z_1 < z_2 < \ldots < z_n$. The closure Θ_{id} of $\Theta^\circ_{\mathrm{id}}$ in $\overline{M}_{0,n+1}(\mathbb{R})$ is simply connected and contains z_{JM}. For any point $z \in \Theta_{\mathrm{id}}$ there is a unique path contained in Θ_{id}, up to homotopy, starting at z and ending at z_{JM}. Define $\mathcal{A}(\square^n, z)_\mu = \mathcal{A}(z)_\mu$.

Corollary 3.2.23. *Let* $z = (z_1, z_2, \ldots, z_n)$ *be an* n*-tuple of real numbers such that* $z_1 < z_2 < \ldots < z_n$*. The Bethe spectrum* $\mathcal{A}(z)_\mu$ *is canonically labelled by standard* μ*-tableaux.*

Proof. For any point $z \in \Theta^\circ_{\mathrm{id}}$, by Corollary 3.1.13, the algebra $G(z)$ has simple spectrum on $S(\mu)$ and is thus maximally commutative in $\mathrm{End}(S(\mu))$. By Theorem 3.1.12, it equals the Bethe algebra $A(z)_\mu$. So, over $\Theta^\circ_{\mathrm{id}}$, the Bethe spectrum π_μ and the Gaudin spectrum $\overline{\pi}_\mu$ are equal. In order to obtain a labelling for $\mathcal{A}(z)_\mu$, we analytically continue the labelling provided by Theorem 3.2.22 (iii) from the point z_{JM}. Since Θ_{id} is simply connected, this is unique. $\square$

3.3 Crystals

Crystals provide an idealised combinatorial model for the representation theory of $\mathfrak{gl}_r$ (and more generally, semisimple Lie algebras). In this thesis, they will turn out to give a combinatorial description of the monodromy of the Bethe spectrum. Crystals were first defined by Kashiwara [KN94; Kas91; Kas90], and the definitions here are special cases of the original definitions. A comprehensive reference for the material is [HK02].

3.3.1 Combinatorial crystals

Definition 3.3.1. A *(finite)* $\mathfrak{gl}_r$*-crystal* is a finite set B along with maps

$$\mathrm{wt} : B \longrightarrow \mathbb{Z}^r,$$
$$e_i, f_i : B \longrightarrow B \sqcup \{0\},$$

for $i = 1, 2, \ldots, r$ obeying the following axioms for any $b, b' \in B$,

(i) if $e_i(b) \neq 0$ then $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$,

(ii) if $f_i(b) \neq 0$ then $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$,

(iii) $b' = e_i(b)$ if and only if $b = f_i(b')$, and

(iv) if $b, b' \in B$ such that $e_i(b) = f_i(b') = 0$ and $f_i^k(b) = b'$ for some $k \geq 0$, then $\mathrm{wt}(b') = s_i \cdot \mathrm{wt}(b)$

where $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$ with the 1 in the i^{th} position and s_i is the transposition swapping the i^{th} and $(i+1)^{\text{th}}$ coordinate of $\mathbb{Z}^r$.

Example 3.3.2. Let $r = 2$. For each pair of nonnegative integers m, n such that $m \geq n$, there is a crystal $\mathtt{B}(m, n) = \{v_0, v_1, \ldots, v_{m-n}\}$. This crystal is defined by $\mathrm{wt}(v_i) = (m - i, n + i)$ and $f_1(v_k) = v_{k+1}$ (by convention we say $v_{m-n+1} = 0$). Note the action of e_1 is completely determined by condition (iii) in Definition 3.3.1. We can depict the crystal by

$$v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_{m-n},$$

where the arrow denote the action of f_1. It isn't hard to see these are the only crystals for which the picture above is connected.

In general for a $\mathfrak{gl}_r$-crystal B, we can draw the coloured, directed graph which vertices labelled by B and edges coloured by i indicating the action of f_i (with no edge if $f_i(b) = 0$). We call this the *crystal graph*. The direct sum of two crystals B and C is defined to be the disjoint union. A crystal is called *irreducible* if its associated crystal graph is connected. An element of a crystal is called a *highest weight element* (or sometimes we will say *highest weight vector*) if it is killed by the operator e_i for all i.

Definition 3.3.3. A *morphism of crystals* $\varphi : B \longrightarrow C$ is given by a function on sets $\varphi : B \sqcup \{0\} \longrightarrow C \sqcup \{0\}$, such that the following conditions hold,

(i) $\varphi(0) = 0$,

(ii) if $\varphi(b) \neq 0$ then $\mathrm{wt}(\varphi(b)) = \mathrm{wt}(b)$,

(iii) we have $e_i \circ \varphi = \varphi \circ e_i$ and $f_i \circ \varphi = \varphi \circ f_i$ (we set $e_i(0) = f_i(0) = 0$).

In Section 3.3.2 we will recall the definition of an irreducible highest weight crystal $\mathtt{B}(\lambda)$ for every $\lambda \in \mathtt{Part}(r)$ (for $r = 2$ we have already done so in Example 3.3.2).

Definition 3.3.4. The category $\mathtt{Crys} = \mathtt{Crys}(\mathfrak{gl}_r)$ of $\mathfrak{gl}_r$-crystals is the category with objects crystals isomorphic to a direct sum of $\mathtt{B}(\lambda)$ and morphisms are morphisms of crystals.

We should think of a crystal as coming from a basis of a finite dimensional $\mathfrak{gl}_r$-module. In fact, in a very precise sense, they are degeneration of a very special *global basis*. One of the properties we would like is to be able to construct a tensor product basis. To this end, for a crystal B define the functions $\varepsilon_i, \phi_i : B \longrightarrow \mathbb{Z}$,

$$\begin{aligned} \varepsilon_i(b) &= \max\{a \mid e_i^a(b) \neq 0\}, \\ \phi_i(b) &= \max\{a \mid f_i^a(b) \neq 0\}. \end{aligned}$$

Definition 3.3.5. Let B and C be $\mathfrak{gl}_r$-crystals. The *tensor product* crystal $B \otimes C$ is defined as the set $B \times C$ with the structure maps

(i) $\mathrm{wt} : B \times C \longrightarrow \mathbb{Z}^r$ defined by $\mathrm{wt}(b \otimes c) = \mathrm{wt}(b) + \mathrm{wt}(c)$,

(ii) $e_i, f_i : B \times C \longrightarrow B \times C$ defined by

$$e_i(b \otimes c) = \begin{cases} e_i(b) \otimes c & \text{if } \phi_i(b) \geq \varepsilon_i(c), \\ b \otimes e_i(c) & \text{if } \phi_i(b) < \varepsilon_i(c), \end{cases}$$

$$f_i(b \otimes c) = \begin{cases} f_i(b) \otimes c & \text{if } \phi_i(b) > \varepsilon_i(c), \\ b \otimes f_i(c) & \text{if } \phi_i(b) \leq \varepsilon_i(c). \end{cases}$$

Example 3.3.6. In the case $r = 2$, for the tensor product $\mathtt{B}(m_1, n_1) \otimes \mathtt{B}(m_2, n_2)$, we can state the rule more simply. Note that for $\mathtt{B}(m, n)$, $\varepsilon_1(v_k) = k$ and $\phi_1(v_k) = m - n - k$.

$$e_1(v_k \otimes v_l) = \begin{cases} v_{k-1} \otimes v_l & \text{if } k + l \leq m_1 - n_1, \\ v_k \otimes v_{l-1} & \text{if } k + l > m_1 - n_1, \end{cases}$$

$$f_1(v_k \otimes v_l) = \begin{cases} v_{k+1} \otimes v_l & \text{if } k + l < m_1 - n_1, \\ v_k \otimes v_{l+1} & \text{if } k + l \geq m_1 - n_1, \end{cases}$$

This rule tells us we have highest weight vectors (those killed by e_1), $v_0 \otimes v_l$ for $l \leq m_1 - n_1$. Each of these highest weight vectors determines a unique connected component. Since $\mathrm{wt}(v_0 \otimes v_l) = (m_1 + m_2 - l, n_1 + n_2 + l)$ we have

$$\mathtt{B}(m_1, n_1) \otimes \mathtt{B}(m_2, n_2) \cong \bigoplus_{l=0}^{\min\{m_1 - n_1, m_2 - n_2\}} \mathtt{B}(m_1 + m_2 - l, n_1 + n_2 + l),$$

which is a special case of the *Littlewood-Richardson rule*.

3.3.2 Irreducible crystals

We give a description of the irreducible $\mathfrak{gl}_r$-crystal $\mathtt{B}(\lambda)$ for every $\lambda \in \mathtt{Part}(r)$. As a set $\mathtt{B}(\lambda) = \mathtt{SSYT}(\lambda)$ where the alphabet we use is $[r]$. This is motivated by the fact that $\dim L(\lambda) = \#\mathtt{SSYT}(\lambda)$. We first define the crystal for $\lambda = (1) = \square$, corresponding to the vector representation $V = L(\square)$.

Definition 3.3.7. The crystal $\mathtt{B} = \mathtt{B}(\square)$ is defined by setting $\mathtt{B} = \mathtt{SSYT}(\square) = [r]$ as sets and defining the structure maps by $\mathrm{wt}(k) = (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in the k^{th} position and $f_i(k) = \delta_{ik}(k+1)$.

The maps $\varepsilon_i, \phi_i : \mathtt{B} \longrightarrow \mathbb{Z}$ are $\epsilon_i(k) = \delta_{k(i+1)}$ and $\phi_i(k) = \delta_{ik}$. The crystal graph associated to $\mathtt{B}$ is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r}.$$

The crystal $\mathtt{B}(\lambda)$ will be constructed as a subcrystal of $\mathtt{B}^{\otimes n}$ where $n = |\lambda|$. First let us give a description of $\mathtt{B}^{\otimes n}$. Let $\mathtt{words}(n)$ be the set of words of length n in the alphabet $[r]$. An element of $\mathtt{B}^{\otimes n}$ is of the form

$$\boxed{x_1} \otimes \boxed{x_2} \otimes \cdots \otimes \boxed{x_n},$$

For integers $1 \leq x_1, x_2, \dots, x_n \leq r$. Thus we can identify the sets $\mathtt{B}^{\otimes n}$ and $\mathtt{words}(n)$. Let $w = x_1 x_2 \cdots x_n \in \mathtt{words}(w)$ be a word of length n. For $i = 1, 2, \dots, r-1$, let $w_i \subset w$ be the subword of letters i and $i+1$. Perform the following mutation on w_i: delete any pair $(i+1)i$ which occur in sequence in w_i then repeat until we are left with a string $\bar{w}_i$ of i's followed by a string of $(i+1)$'s. Thus $\bar{w}_i = x_{j_1} x_{j_2} \cdots x_{j_k}$ for some $1 \leq j_1 < j_2 < \ldots < j_k \leq n$. Let a be the integer such that $x_{j_a} = i$ and $x_{j_{a+1}} = i+1$ (in the case when $\bar{w}_i$ contains no i's we let $a = 0$). Define the *i-signature* of w to be $\mathrm{sgn}_i(w) = (j_a, j_{a+1})$ where $j_l = \infty$ when $l < 1$ or $l > k$.

Example 3.3.8. Suppose $w = 31422312242$. Then

$$w_1 = 1221222 = x_2 x_4 x_5 x_7 x_8 x_9 x_{11}$$

and $\bar{w}_1 = 12222 = x_2 x_4 x_8 x_9 x_{11}$ so $\mathrm{sgn}_1(w) = (2, 4)$. Now $w_2 = 3223222$ which is $x_1 x_4 x_5 x_6 x_8 x_9 x_{11}$ and $\bar{w}_2 = 222 = x_5 x_9 x_{11}$ so $\mathrm{sgn}_2(w) = (11, \infty)$.

Proposition 3.3.9. *Let $w = x_1 x_2 \cdots x_n \in \mathtt{words}(n)$ and $\mathrm{sgn}_i(w) = (a, b)$. The crystal structure on $\mathtt{B}^{\otimes n} = \mathtt{words}(n)$ is given by*

$$f_i(w) = \begin{cases} x_1 x_2 \cdots f_i(x_a) \cdots x_n & \textit{if } a \neq \infty, \\ 0 & \textit{otherwise}, \end{cases}$$

$$e_i(w) = \begin{cases} x_1 x_2 \cdots e_i(x_b) \cdots x_n & \textit{if } b \neq \infty, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. See [HK02, Section 4.4]. □

Example 3.3.10. Continuing Example 3.3.8 we obtain

$$\begin{aligned} f_1(w) &= 3f_1(1)422312242 = 32422312242 \\ e_1(w) &= 314e_1(2)2312242 = 31412312242 \\ f_2(w) &= 3142231224f_2(2) = 31422312243 \\ e_2(w) &= 0 \end{aligned}$$

Fix a standard tableau $R \in \mathtt{SYT}(\lambda)$. With this choice we can define an embedding of $\mathtt{SSYT}(\lambda)$ into the set $\mathtt{words}(n)$,

$$\mathtt{read}_R = \mathtt{RSK}^{-1}(-, R) : \mathtt{SSYT}(\lambda) \longrightarrow \mathtt{words}(n).$$

Theorem 3.3.11. *The following properties are satisfied by* $\mathtt{read}_R$*:*

(i) The map $\mathtt{read}_R$ *is injective and its image in* $\mathtt{words}(n) = \mathtt{B}^{\otimes n}$ *is stable under the operators* e_i *and* f_i*.*

(ii) For two standard tableaux $R, R' \in \mathtt{SYT}(\lambda)$*, the composition* $\mathtt{read}_{R'} \circ \mathtt{read}_R^{-1}$ *is an isomorphism of crystals.*

(iii) Given a word $w \in \mathtt{words}(n)$*, there exists some* λ *and* $S \in \mathtt{SSYT}(\lambda)$ *such that* $\mathtt{read}_R(S) = w$*.*

Proof. Injectivity follows by Theorem 3.2.18 which asserts the RSK correspondence is bijective. In order to check the image of $\mathtt{read}_R$ is stable under the crystal operators, we need show that $\mathtt{Q}(e_i(w)) = \mathtt{Q}(f_i(w)) = R$. By construction, the recording tableau of a word w depends only on the relative sizes of its letters (recall the convention set in Remark 3.2.2 that if we have two letters both equal to i, then we consider the one to the left to be smaller). If we apply f_i to our word, the only time it might change the relative size of the letters is if we have a situation of the following form,

$$f_i(w) = f_i(\cdots ii \cdots) = \cdots f_i(i)i \cdots,$$

However by Proposition 3.3.9, such a situation can never happen. Thus part (i) is proven.

To prove (ii), if two words w and w' are related by $w' = \mathtt{read}_{R'} \circ \mathtt{read}_R^{-1}(w)$ then they must be slide equivalent. By Proposition 3.2.8, two words are slide equivalent if and only if they differ by a sequence of Knuth moves. In fact the crystal operators commute with the Knuth moves. (This is a well known fact whose proof is elementary but involved, see [Lot02, Theorem 5.5.1] for a proof.) Furthermore, $\mathrm{wt}(w)$ is simply the sequence which counts the number of 1's, number of 2's and so on. This is also preserved by slide equivalence. Thus we have a crystal morphism and since it is a bijection it is an isomorphism. For part (iii), we need to find a semistandard tableaux S, slide equivalent to w. We can take $S = \mathtt{Rect}(w)$ for this purpose. □

Definition 3.3.12. We define the crystal $\mathtt{B}(\lambda)$ as the crystal structure on $\mathtt{SSYT}(\lambda)$ induced by the map $\mathtt{read}_R$ for any $R \in \mathtt{SYT}(\lambda)$.

Theorem 3.3.11 tells us that $\mathtt{B}(\lambda)$ is well defined and that it does not depend on the choice of $R \in \mathtt{SYT}(\lambda)$. As a direct consequence of the theorem we have a crystal isomorphism afforded by the RSK correspondence,

$$\mathtt{B}^{\otimes n} \longrightarrow \bigsqcup_{|\lambda|=n} \mathtt{B}(\lambda) \times \mathtt{SYT}(\lambda). \tag{3.3.1}$$

The *Yamanouchi tableau*, $\mathtt{Y}(\lambda)$ of shape λ is the semistandard tableau with its k^{th} row filled with the integer k.

Proposition 3.3.13. *The crystal* $\mathtt{B}(\lambda)$ *is irreducible, with a unique element* $\mathtt{Y}(\lambda)$ *of highest weight* λ.

Proof. See [HK02, Theorem 7.4.1]. $\square$

Proposition 3.3.14. *Let* $A, B \in \mathtt{Crys}$*, then* $A \otimes B \in \mathtt{Crys}$*. Thus* $\mathtt{Crys}$ *is a monoidal category.*

Proof. First, by (3.3.1), the crystal $\mathtt{B}^{\otimes n}$ is in $\mathtt{Crys}$. For more general tensor products it is enough to prove the proposition for A and B irreducible crystals. Let $A = \mathtt{B}(\mu)$ and $B = \mathtt{B}(\lambda)$ and let $m = |\mu|$ and $n = |\lambda|$. Then by construction, we can identify A with a subcrystal of $\mathtt{B}^{\otimes m}$ and B with a subcrystal of $\mathtt{B}^{\otimes n}$ and thus $A \otimes B$ with a subcrystal of $\mathtt{B}^{m} \otimes \mathtt{B}^{\otimes n} \cong \mathtt{B}^{\otimes(m+n)}$. By Theorem 3.3.11 (iii) every irreducible subcrystal of $\mathtt{B}^{\otimes(m+n)}$, and thus of $A \otimes B$, is isomorphic to $\mathtt{B}(\nu)$ for some $\nu \in \mathtt{Part}_{n+m}$. Hence $A \otimes B \in \mathtt{Crys}$. $\square$

Proposition 3.3.15. *Suppose we have a nonzero morphism* $\varphi : \mathtt{B}(\lambda) \longrightarrow \mathtt{B}(\mu)$ *of crystals, then it is an isomorphism.*

Proof. The morphism φ must take the highest weight element to to highest weight element. Since these must have the same weight we must have isomorphic crystals. $\square$

3.3.3 Coboundary monoidal categories

Although crystals inherit a lot of the combinatorial structure of $\mathfrak{gl}_r$-modules and their tensor products, one big difference is crystals do not form a braided monoidal category. Whilst $\mathtt{Crys}$ does not have a braided structure, the tensor product still makes it a monoidal category and we still have natural isomorphims $B \otimes C \longrightarrow C \otimes B$ obeying a slightly different coherence relation. This is called a *coboundary category.*

Definition 3.3.16. A monoidal category $\mathcal{C}$ is *coboundary* if there exists a commutor $\sigma^c_{A,B} : A \otimes B \longrightarrow B \otimes A$ for objects A, B in $\mathcal{C}$ that satisfies the following conditions.

(i) That $\sigma^c_{B,A} \circ \sigma^c_{A,B} = 1$, and

(ii) that for any objects A, B, C, σ^c obeys the *cactus relation*:

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{\sigma^c_{A,B} \otimes \mathrm{id}} & B \otimes A \otimes C \\ \downarrow{\scriptstyle \mathrm{id} \otimes \sigma^c_{B,C}} & & \downarrow{\scriptstyle \sigma^c_{B\otimes A,C}} \\ A \otimes C \otimes B & \xrightarrow{\sigma^c_{A,C\otimes B}} & C \otimes B \otimes A \end{array} \tag{3.3.2}$$

A commutor obeying the above conditions will be called a *cactus commutor*. Just as the structure of a braiding on a monoidal category induces an action of the braid group on n-fold tensor products of objects, coboundary categories induce a similar structure. For objects $A_1, A_2, \dots, A_n$, for $1 \leq p < q \leq n$ set

$$\sigma_{pq} = \mathrm{id}^{\otimes(p-1)} \otimes \sigma^c_{A_p \otimes \cdots \otimes A_{q-1}, A_q} \otimes \mathrm{id}^{\otimes(n-q)}.$$

Then define $s_{p(p+1)} = \sigma_{p(p+1)}$ and inductively $s_{pq} = s_{(p+1)q} \circ \sigma_{pq}$. We should think of s_{pq} as swapping the order of $A_p, A_{p+1}, \dots, A_q$.

Proposition 3.3.17 ([HK06, Lemma 3 and Lemma 4]). *For $1 \leq p < q \leq n$ define the following elements of the symmetric group.*

$$\hat{s}_{pq} = \begin{pmatrix} 1 & 2 & \cdots & p-1 & p & p+1 & \cdots & q & \cdots & n \\ 1 & 2 & \cdots & p-1 & q & q-1 & \cdots & p & \cdots & n \end{pmatrix}.$$

The isomorphisms s_{pq} obey the following relations.

(i) $s_{pq}^2 = 1$

(ii) $s_{pq} \circ s_{kl} = s_{kl} \circ s_{pq}$ *if the intervals $[p, q]$ and $[k, l]$ are disjoint.*

(iii) $s_{pq} \circ s_{kl} = s_{mn} \circ s_{pq}$ *if $[k, l] \subseteq [p, q]$, where $n = \hat{s}_{pq}(k)$ and $m = \hat{s}_{pq}(l)$.*

Definition 3.3.18. The group generated by formal symbols s_{pq}, that obey the relations in Propostion 3.3.17 is called *n-fruited cactus group*. We denote this group J_n.

Note that the permutations $\hat{s}_{pq}$ also obey these relations, so the map $s_{pq} \mapsto \hat{s}_{pq}$ is a group homomorphsim. We call the kernel of this map the *pure cactus group*, PJ_n. We have an exact sequence

$$1 \longrightarrow PJ_n \longrightarrow J_n \longrightarrow S_n \longrightarrow 1. \tag{3.3.3}$$

Remark 3.3.19. From the definition, it is clear that the pure cactus group acts on arbitrary tensor products of objects in a coboundary category and not just on n-fold tensor products of a single object.

3.3.4 Coboundary structure on $\mathtt{Crys}(\mathfrak{gl}_r)$

Given a crystal $B \in \mathtt{Crys}$. By definition, the irreducible components of B are isomorphic to $\mathtt{B}(\lambda)$ for some λ. Thus we can identify the elements of B with semistandard tableaux. In particular we are able to perform the Schützenberger involution on elements of B.

Definition 3.3.20. For two crystals, $A, B \in \mathtt{Crys}$ define the following map

$$\sigma^c_{A,B} : A \otimes B \longrightarrow B \otimes A, \quad \text{where } \sigma^c_{A,B}(a \otimes b) = \xi(\xi(b) \otimes \xi(a)). \tag{3.3.4}$$

Theorem 3.3.21 ([HK06, Theorem 2])**.** *The maps $\sigma^c_{A,B}$ are natural isomorphisms of crystals and give the category of crystals $\mathtt{Crys}(\mathfrak{gl}_r)$ a coboundary structure.*

3.3.5 Cactus group actions

In this section we will give a description on the action of J_n on the crystal $\mathtt{B}^{\otimes n}$. By (3.3.1) we have a decomposition into irreducible crystals,

$$\mathtt{B}^{\otimes n} \cong \bigsqcup_{|\lambda|=n} \mathtt{B}(\lambda)^{\oplus d_\lambda}. \tag{3.3.5}$$

where $d_\lambda = \#\mathtt{SYT}(\lambda)$. Any isomorphism of $\mathtt{B}^{\otimes n}$ must send each summand $\mathtt{B}(\lambda)$ to another copy of $\mathtt{B}(\lambda)$ by Proposition 3.3.15. Identifying the crystal $\mathtt{B}^{\otimes n}$ with the set $\bigsqcup_\lambda \mathtt{SSYT}(\lambda) \times \mathtt{SYT}(\lambda)$, we see that any isomorphism must fix $\mathtt{SSYT}(\lambda)$ and permute the elements of $\mathtt{SYT}(\lambda)$. A small example may help clarify this.

Example 3.3.22. Consider the case $n = r = 3$. The crystal $\mathtt{B}^{\otimes 3}$ can be computed using the tensor pruduct rule and identified with pairs via the RSK correspondence to produce the diagram in Figure 3.2. So if we have an isomorphism of $\mathtt{B}^{\otimes 3}$ it must fix those components labelled by tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

as it would have to send each of these to a standard tableaux of the same shape (of which there is only one in each case). However the two remaining connected components can be permuted, but this must preserve the semistandard tableaux. Hence we have exactly two automorphisms of $\mathtt{B}^{\otimes 3}$, the identity and the isomorphism swapping

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

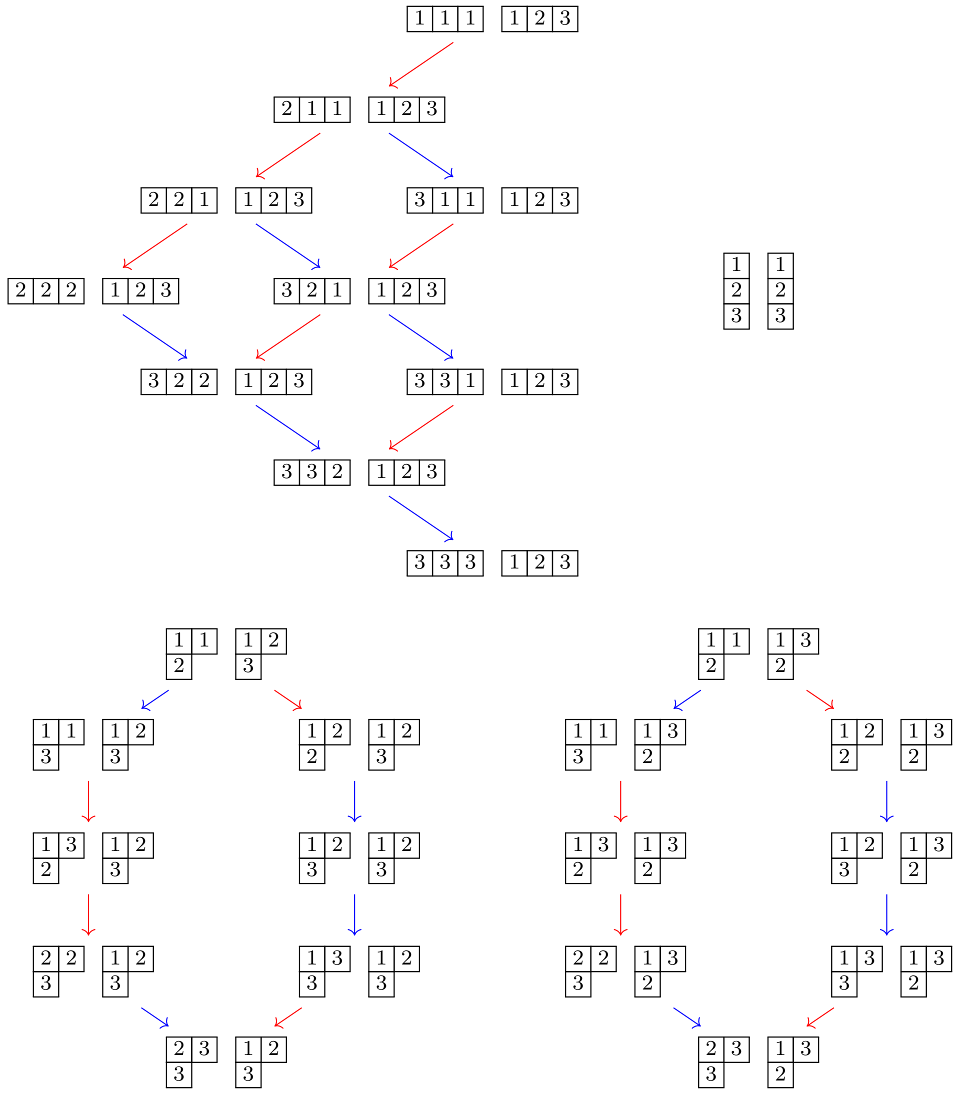


Figure 3.2: The crystal graph $\mathtt{B}^{\otimes 3}$

Proposition 3.3.23. *Suppose Φ is an automorphism of $\mathtt{B}^{\otimes n}$ which we identify with the set $\bigsqcup_\lambda \mathtt{SSYT}(\lambda) \times \mathtt{SYT}(\lambda)$. Then $\Phi(P, Q) = (P, \phi(Q))$ for some permutation ϕ of the set $\mathtt{SYT}(\lambda)$ where λ is the shape of Q. Furthermore every automorphism arises in this way.*

We would like to understand the action of J_n on $\mathtt{B}^{\otimes n}$. Proposition 3.3.23 tells us that this action induces an action of J_n on standard tableaux and furthermore is completely determined by it. By the definition of the cactus commutor we can express the effect of σ^c_{pq} on a word $w = x_1 x_2 \dots x_n \in \mathtt{B}^{\otimes n}$ using the RSK correspondence and the Schützenberger involution.

Lemma 3.3.24. *On the word w, the Schützenberger involution is given by $\mathtt{RSK}^{-1}(\xi(P), Q)$ where $P = \mathtt{P}(w)$ and $Q = \mathtt{Q}(w)$, and ξ is the usual Schützenberger involution.*

Proof. To apply the Schützenberger involution to w, we need to identify the irreducible component of $\mathtt{B}^{\otimes n}$ containing w with an irreducible crystal $\mathtt{B}(\lambda)$. By Definition 3.3.12, we do this by applying $\mathtt{read}_Q^{-1}$, applying the Schützenberger involution and then applying $\mathtt{read}_Q$. So

$$\xi(w) = \mathtt{read}_Q \circ \xi \circ \mathtt{read}_Q^{-1}(w) = \mathtt{RSK}^{-1}(-, Q) \circ \xi \circ \mathtt{P}(w) = \mathtt{RSK}^{-1}(\xi(P), Q). \qquad \square$$

Thus applying this result to the definition of the crystal commutator produces,

$$\sigma^c_{pq}(w) = x_1 x_2 \cdots x_{p-1}\, \xi(\xi(x_q)\, \xi(x_p) \cdots \xi(x_{q-1}))\, x_{q+1} \cdots x_N.$$

Lemma 3.3.25. *Suppose $q = p+1$, then $s_{pq} = \sigma^c_{pq} = \mathrm{id}$.*

Proof. Suppose $1 \le x \le r$ is an integer. Recall $x^* = r - x + 1$, and by definition $\xi(x) = x^*$. This means, for any two integers $1 \le a, b \le r$,

$$\sigma^c_{12}(ab) = \xi(\xi(b)\xi(a)) = \xi(b^* a^*).$$

Suppose that $a \le b$ so $a^* \ge b^*$. Then $\mathtt{P}(b^* a^*) = \boxed{b^*}\boxed{a^*}$ and so $\sigma^c_{12}(ab) = \xi(b^*, a^*) = ab$. $\square$

Lemma 3.3.26. *Suppose that $q = p+2$, then the action of $s_{pq} = \sigma^c_{pq}$ on the triple $x_p x_{p+1} x_{p+2}$ is given by the Knuth moves (we set $x_p = x, x_{p+1} = y, x_{p+2} = z$ for clarity):*

$$xyz \mapsto xzy \quad \text{if } z < x \le y \text{ or } y < x \le z,$$
$$xyz \mapsto yxz \quad \text{if } x \le z < y \text{ or } y \le z < x.$$

Proof. $s_{13} = \sigma^c_{23} \circ \sigma^c_{13}$ but by Lemma 3.3.25, σ^c_{23} acts trivially, so we need to calculate $\sigma^c_{13}(xyz)$. In the case $x \le y \le z$, $\sigma^c_{13}(xyz) = \xi(\xi(yz)\xi(x))$. $\xi(x) = x^*$ and $\mathtt{P}(yz) = \boxed{y}\boxed{z}$ so $\xi(yz) = z^* y^*$. Then $\sigma^c_{13}(xyz) = \xi(z^* y^* x^*)$ but $\mathtt{P}(z^* y^* x^*) = \boxed{z^*}\boxed{y^*}\boxed{x^*}$ so $\xi(z^* y^* x^*) = xyz$. Hence the action of s_{13} is trivial. The remaining cases following similarly. $\square$

As can be seen from Lemma 3.3.26, even when $q = p+2$, the action of s_{pq} can be quite complicated, however we have enough information to work out the action of J_3 on the crystal $\mathtt{B}^{\otimes 3}$.

Example 3.3.27. Because we know the action is determined by its induced action on standard tableaux, we only need to check what it does to the vertices

$$\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right), \text{ and } \left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right).$$

By Proposition 3.3.15 we know the action on the first two vertices must be trivial. We also know that $s_{12}, s_{23} \in J_3$ act trivially by Lemma 3.3.25. To calculate the action on the remaining vertices we note

$$\mathtt{RSK}^{-1}\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right) = 132, \text{ and } \mathtt{RSK}^{-1}\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right) = 312.$$

Using Lemma 3.3.26 we see that $s_{13}(132) = 312$ and $s_{13}(312) = 132$. Hence s_{13} swaps the two connected components representing the irreducible $\mathtt{B}(2,1)$.

3.4 Main results

In this section we will outline the main results proved in the remainder of this thesis, including the important special case of the n-fold tensor product of the vector representation.

3.4.1 Monodromy of the Bethe spectrum

The first main result of this thesis, proved in Section 5.4, is to identify the monodromy of the Bethe spectrum with the action of the cactus group on tensor products of crystals. We can consider the action of PJ_n on

$$\mathtt{B}(\lambda_\bullet) = \mathtt{B}(\lambda_1) \otimes \mathtt{B}(\lambda_2) \otimes \cdots \otimes \mathtt{B}(\lambda_n),$$

for a sequence of partitions $\lambda_\bullet$. Since the elements of PJ_n commute with the operators e_i and commute with the weight function wt, we can restrict to an action of PJ_n on

$$\mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}} = [\mathtt{B}(\lambda_1) \otimes \mathtt{B}(\lambda_2) \otimes \cdots \otimes \mathtt{B}(\lambda_n)]_\mu^{\mathsf{sing}}.$$

Recall the Bethe spectrum associated to the data $(\lambda_\bullet, \mu)$ is a map

$$\pi_{\lambda_\bullet,\mu} : \mathcal{A}(\lambda_\bullet, \mu) \longrightarrow M_{0,n+1}(\mathbb{C}).$$

We denote the fibre over a point $z \in M_{0,n+1}(\mathbb{C})$ by $\mathcal{A}(\lambda_\bullet, z)_\mu$.

Theorem A. *There exists a homomorphism $PJ_n \to \mathrm{Gal}(\pi_{\lambda_\bullet,\mu})$ from the pure cactus group to the Galois group of $\pi_{\lambda_\bullet,\mu}$ and a bijection*

$$\mathcal{A}(\lambda_\bullet, z)_\mu \longrightarrow \mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}},$$

equivariant for the induced action of PJ_n.

The strategy of the proof is as follows. Using a result of Speyer [Spe14], there is an extension of $\pi_{\lambda_\bullet,\mu}$ to the entire compact moduli space $\overline{M}_{0,n+1}(\mathbb{C})$ defined using Schubert calculus. This extension is an unramified covering over $\overline{M}_{0,n+1}(\mathbb{R})$, and we will be able to calculate its monodromy combinatorially. We will see in Section 4.5.2 that $\pi_1(\overline{M}_{0,n+1}(\mathbb{R})) = PJ_n$, and thus gives us the map to the Galois group. The identification with highest weight elements of a crystal then reduces to a combinatorial statement, which we prove in Section 5.3.5.

3.4.2 Identification of labelling sets

In the important special case when $\lambda_\bullet = (\square^n)$, we have the decomposition (3.3.1) of $\mathtt{B}^{\otimes n}$ into irreducible crystals given by the RSK algorithm,

$$\mathtt{B}^{\otimes n} = \bigsqcup_{|\lambda|=n} \mathtt{B}(\lambda) \times \mathtt{SYT}(\lambda),$$

Proposition 3.3.13 says $\mathtt{B}(\lambda)$ is a highest weight crystal with unique highest weight element $\mathtt{Y}(\lambda)$. Thus we see that $[\mathtt{B}^{\otimes n}]_\mu^{\mathsf{sing}} = \{(\mathtt{Y}(\mu), S) \mid S \in \mathtt{SYT}(\mu)\}$ and so we can label the elements canonically by standard μ-tableaux.

Let $z = (z_1, z_2, \dots, z_n)$ be an n-tuple of distinct real numbers in increasing order, i.e. such that $z_1 < z_2 < \ldots < z_n$. Recall from Corollary 3.2.23 that the set $\mathcal{A}(z)_\mu$ is also labelled by standard μ-tableaux. The way this labelling is calculated in practice is to take a point $\chi \in \mathcal{A}(z)_\mu$ and choose any path from z to z_{JM}. We think of χ as a functional $A(z)_\mu \to \mathbb{C}$ and we take the limit of $\chi(z_a H_a(z))$ as $z_a \to \infty$ such that $z_a/z_{a+1} \to 0$. By Proposition 2.4.7, this limit identifies a simultaneous eigenspace for the Jucys-Murphy elements and we know from Theorem 3.2.22 that $\lim_{z\to\infty} \chi(z_a H_a(z)) = c_s(a)$ for some standard μ-tableau S.

Let z be as above, real and in increasing order. Theorem A provides a bijection between points in the fibre of the Bethe spectrum and highest weight vectors of weight μ in the crystal,

$$\mathbb{X}_\mu(z) : \mathcal{A}(z)_\mu \longrightarrow [\mathtt{B}^{\otimes n}]_\mu^{\mathsf{sing}} = \{\mathtt{Y}(\mu)\} \times \mathtt{SYT}(\mu) = \mathtt{SYT}(\mu).$$

The following theorem says this bijection is given by the same process of taking a limit and considering the eigenvalues as above, thus it identifies the labelling of both sets by standard μ-tableaux

Theorem B. *For $z = (z_1, z_2, \dots, z_n)$ an n-tuple of distinct real numbers such that $z_1 < z_2 < \ldots < z_n$, the bijection $\mathbb{X}_\mu(z) : \mathcal{A}(z)_\mu \longrightarrow \mathtt{SYT}(\mu)$ is given by $\mathbb{X}_\mu(z)(\chi) = S$, where S is the unique tableau with*

$$c_S(a) = \lim_{z\to\infty} \chi_S(z_a H_a(z)).$$

Critical points of the master function and their relation to the Bethe spectrum will be recalled in Section 4.8. The asymptotics of critical points as described by Reshetikhin and Varchenko [RV95] give a labelling of the critical points at z by standard μ-tableaux as shown by Marcus [Mar10]. We show this labelling is compatible with the combinatorial description of Speyer's covering and this will be enough to prove the theorem.

Chapter 4

Schubert calculus and the Bethe Ansatz

Mukhin, Tarasov and Varchenko [MTV09a], described a strong link between the Bethe algebras introduced in Section 3.1.3 and intersections of Schubert cells. In this chapter we recall definitions and results about Schubert calculus. In particular we recall the definition of the *osculating flag* associated to a complex number. Intersections of Schubert varieties with respect to osculating flags at distinct complex numbers are generically reduced and always have the expected dimension, this is a result of Eisenbud and Harris [EH83]. In fact, using the connection to Bethe algebras, Mukhin, Tarasov and Varchenko [MTV09b] proved a conjecture of Shapiro and Shapiro; Schubert intersections with respect to osculating flags at distinct real points are a reduced union of real points (c.f. Theorem 3.1.10). There is a family over $M_{0,n+1}(\mathbb{C})$ of such Schubert intersections and in Section 4.7 we will show how to use the results of Mukhin, Tarasov and Varchenko to show this family is isomorphic to the Bethe spectrum. Speyer [Spe14] constructed a compactification of this family of Schubert intersections over $\overline{M}_{0,n+1}(\mathbb{C})$ and gave a combinatorial description of the restriction to $\overline{M}_{0,n+1}(\mathbb{R})$, we recall the definitions in Sections 4.2 and 4.5. This combinatorial description will be the main technical tool we will use to relate the monodromy of this family to automorphisms of crystals.

4.1 Schubert varieties

In this section we recall some definitions and facts from Schubert calculus. Let E be a d-dimensional complex vector space and let $\mathrm{Gr}(r, E)$ denote the *Grassmanian variety* of r-dimensional subspaces. Fix a complete flag $\mathcal{F}_\bullet$,

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_d = E$$

of subspaces of E. Recall $\texttt{Part}(r,d)$ is the set of partitions with at most r rows and at most $d-r$ columns and let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}) \in \texttt{Part}(r,d)$. Define the *jump sequence*

$$1 \leq i_1 < i_2 < \ldots < i_r \leq d,$$

associated to λ by $i_k = \lambda^{(r-k+1)} + k$.

Definition 4.1.1. The *Schubert cell* for the flag $\mathcal{F}_\bullet$ for the partition λ is the subvariety $\Omega^\circ(\lambda; \mathcal{F}_\bullet)$ consisting of all $X \in \mathrm{Gr}(r,E)$ such that

$$\dim(X \cap \mathcal{F}_{i_k}) = k, \text{ and}$$
$$\dim(X \cap \mathcal{F}_{i_k - 1}) = k-1.$$

The *Schubert variety* $\Omega(\lambda; \mathcal{F}_\bullet)$ is the closure of $\Omega^\circ(\lambda; \mathcal{F}_\bullet)$ in $\mathrm{Gr}(r,E)$.

4.1.1 Osculating flags

All the Grassmannians we consider will be defined relative to some genus 0 smooth curve C. To set this up, choose a very ample line bundle $\mathcal{L}$ on C of degree $d-1$. The *Veronese embedding* is the morphism

$$\varepsilon : C \longrightarrow \mathbb{P}H^0(C, \mathcal{L})^*,$$

where a point p is sent by ε to the hyperplane of sections vanishing at p. Let

$$\mathrm{Gr}(r,d)_C = \mathrm{Gr}(r, H^0(C, \mathcal{L})).$$

Define the r^{th} *associated curve* $\varepsilon_r : C \longrightarrow \mathrm{Gr}(r,d)_C$ which sends a point p to the space of sections vanishing to order at least $d-r$ at p. That is

$$\varepsilon_r(p) = H^0(C, \mathcal{I}_p^{d-r} \otimes \mathcal{L}) \subset H^0(C, \mathcal{L}),$$

Here $\mathcal{I}_p$ is the ideal sheaf at the point p. With this notation $\varepsilon = \varepsilon_{d-1}$.

Definition 4.1.2. The flag $\mathcal{F}_\bullet(p)$ on $H^0(C,\mathcal{L})^*$, defined by $\mathcal{F}_i(p) = \varepsilon_i(p)$ is called the *osculating flag* at p.

Example 4.1.3. We can make this concrete by considering the case $C = \mathbb{P}^1$. Fix the standard homogeneous coordinates $[x : y]$ on $\mathbb{P}^1$. Choose the line bundle $\mathcal{O}_{\mathbb{P}^1}(d-1)$. Then $H^0(\mathbb{P}^1, \mathcal{O}(d-1)) = \mathbb{C}[x,y]_{d-1}$, the homogeneous polynomials of degree $d-1$. If we work in the affine patch where $y \neq 0$ then we identify identify this with $\mathbb{C}_d[u]$, the space of polynomials of degree *strictly less than* d (u is the coordinate on this patch). The Grassmannian $\mathrm{Gr}(r,d)_{\mathbb{P}^1}$ is then the set of r-dimensional subspaces of $\mathbb{C}_d[u]$. The map ε_r sends the point $[b:1] \in \mathbb{P}^1$ to the subspace $(u-b)^{d-r}\mathbb{C}_r[u]$ and the osculating flag $\mathcal{F}_\bullet(b)$ is

$$(u-b)^{d-1}\mathbb{C}_1[u] \subset (u-b)^{d-2}\mathbb{C}_2[u] \subset \ldots \subset (u-b)\mathbb{C}_{d-1}[u] \subset \mathbb{C}_d[u].$$

The flag $\mathcal{F}_\bullet(\infty)$ is

$$\mathbb{C}_0[u] \subset \mathbb{C}_1[u] \subset \ldots \subset \mathbb{C}_{d-1}[u] \subset \mathbb{C}_d[u].$$

Remark 4.1.4. In the situation of Example 4.1.3 we will drop the subscript $\mathbb{P}^1$ and write $\mathrm{Gr}(r,d)$ instead of $\mathrm{Gr}(r,d)_{\mathbb{P}^1}$.

Suppose we have pairs $(C, \mathcal{L})$ and $(D, \mathcal{K})$ of curves and very ample line bundles as well as an isomorphism $\phi : C \longrightarrow D$ such that $\phi_* \mathcal{L} \cong \mathcal{K}$. We would like some relation between the Grassmannian and osculating flags on each curve. It is important to note it is not possible to choose a canonical isomorphism between $\phi_*\mathcal{L}$ and $\mathcal{K}$, however we have the following fact.

Lemma 4.1.5. *Let $\mathcal{E}$ be an invertible $\mathcal{O}_X$-module for a projective $\mathbb{C}$-scheme X. Then* $\mathrm{End}(\mathcal{E}) \cong \mathbb{C}$.

Proof. Note that $\mathrm{End}(\mathcal{O}_X) \cong \mathbb{C}$. The lemma follows from the fact $\mathcal{O}_X \cong \mathcal{E} \otimes \mathcal{E}^*$ and the hom-tensor adjunction formula:

$$\begin{aligned}
\mathbb{C} \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) &\cong \mathrm{Hom}(\mathcal{E} \otimes \mathcal{E}^*, \mathcal{O}_X) \\
&\cong \mathrm{Hom}(\mathcal{E}, \mathcal{H}om(\mathcal{E}^*, \mathcal{O}_X)) \\
&\cong \mathrm{Hom}(\mathcal{E}, \mathcal{E}).
\end{aligned}$$

□

This means the isomorphism $\phi_*\mathcal{L} \cong \mathcal{K}$ is unique up to scalar multiple. Since $H^0(C, -) = H^0(D, \phi_* -)$, we have a canonical induced isomorphism

$$\phi_1 : \mathbb{P}H^0(C, \mathcal{L}) \longrightarrow \mathbb{P}H^0(D, \mathcal{K}),$$

as well as canonical isomorphisms

$$\phi_r : \mathrm{Gr}(r,d)_C \longrightarrow \mathrm{Gr}(r,d)_D$$

for any r.

Lemma 4.1.6. *The isomorphism ϕ_r preserves the associated curves, more precisely, $\phi_r \circ \varepsilon_r = \varepsilon_r \circ \phi$. In particular $\mathcal{F}_i(\phi(p)) = \phi_r(\mathcal{F}_i(p))$.*

Proof. First, choose an isomorphism $\psi : \phi_*\mathcal{L} \longrightarrow \mathcal{K}$. We thus obtain an isomorphism $H^0(C, \mathcal{L}) \longrightarrow H^0(D, \mathcal{K})$ which we also denote by ψ. Let $p \in C$ and let $q = \phi(p)$. We need to show the image of $H^0(C, \mathcal{I}_p^{d-r} \otimes \mathcal{L})$ under ψ is $H^0(D, \mathcal{I}_q^{d-r} \otimes \mathcal{K})$. This follows since ψ is a module homomorphism and thus sends $\phi_*(\mathcal{I}_p^{d-r} \otimes \mathcal{L})$, considered as a submodule of $\phi_*(\mathcal{L})$, to $\mathcal{I}_q^{d-r} \otimes \mathcal{K}$. □

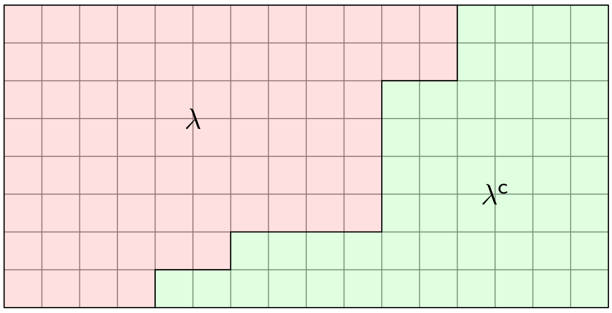


Figure 4.1: The complementary partition

4.1.2 Schubert intersections

For a partition $\lambda \in \mathtt{Part}(r,d)$ and a point $p \in C$ we will denote the Schubert cell corresponding to the osculating flag $\mathcal{F}_\bullet(p)$ by $\Omega^\circ(\lambda;p)_C$ and the corresponding Schubert variety by $\Omega(\lambda;p)_C$. Let λ^c be the partition *complementary* to λ for $\mathrm{Gr}(r,d)_C$. That is, if $\Lambda_{r,d}$ is the rectangular partition with r rows and $d-r$ columns, then λ^c is obtained from $\Lambda_{r,d}\backslash\lambda$ by rotating 180 degrees. This is depicted in Figure 4.1.

Lemma 4.1.7. *Let $(C,\mathcal{L}),(D,\mathcal{K})$ and ϕ_r be as in Section 4.1.1. The image of $\Omega(\lambda;p)_C$ under ϕ_r is $\Omega(\lambda;\phi(p))_D$.*

Proof. Choose an isomorphism $\psi : \phi_*\mathcal{L} \longrightarrow \mathcal{K}$. If $\mathcal{F}$ is a flag in $H^0(C,\mathcal{L})$ then for any subspace $V \subset H^0(C,\mathcal{L})$

$$\dim V \cap \mathcal{F}_i = \dim \psi(V) \cap \psi(\mathcal{F}_i).$$

Thus $\psi_r(\Omega(\lambda,\mathcal{F})_C) = \Omega(\lambda,\psi(\mathcal{F}))$. By Lemma 4.1.6, $\psi(\mathcal{F}(p)) = \mathcal{F}(\phi(p))$. □

Let $k \geq 3$. Given a point $z = (z_1, z_2, \ldots, z_n) \in (\mathbb{P}^1)^k$ and a sequence of partitions $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with each $\lambda_i \in \mathtt{Part}(r,d)$, consider the intersection of Schubert varieties

$$\Omega(\lambda_\bullet; z) = \Omega(\lambda_1; z_1) \cap \Omega(\lambda_2; z_2) \cap \ldots \cap \Omega(\lambda_k; z_k).$$

Let $\Omega'(\lambda_\bullet)$ be the subvariety of $\mathrm{Gr}(r,d) \times (\mathbb{P}^1)^k$ whose fibre over $z \in (\mathbb{P}^1)^k$ is $\Omega(\lambda_\bullet; z)$. An element $\phi \in \mathrm{PGL}_2$ acts on $\mathrm{Gr}(r,d) \times (\mathbb{P}^1)^k$ by acting diagonally on $(\mathbb{P}^1)^k$ and by ϕ_r on $\mathrm{Gr}(r,d)$. Since $k \geq 3$ the action of PGL_2 is free.

Lemma 4.1.8. *The PGL_2 action on $\mathrm{Gr}(r,d) \times (\mathbb{P}^1)^k$ preserves $\Omega'(\lambda_\bullet)$.*

Proof. By Lemma 4.1.7, an element $\phi \in \mathrm{PGL}_2$ maps $\Omega(\lambda_i, z_i)$ isomorphically onto $\Omega(\lambda_i; \phi(z_i))$. Thus the fibre over $z \in (\mathbb{P}^1)^k$, the variety $\Omega(\lambda_\bullet; z)$ is mapped isomorphically onto $\Omega(\lambda_\bullet; \phi(z))$. □

Let $\Delta = \{z \in (\mathbb{P}^1)^k \,|\, z_i \neq z_j \text{ if } i \neq j\}$. Restrict the family $\Omega'(\lambda_\bullet)$ to this open subvariety $(\mathbb{P}^1)^k - \Delta \subset (\mathbb{P}^1)^k$ to obtain a family $\Omega'(\lambda_\bullet)|_{(\mathbb{P}^1)^k-\Delta}$. Note the PGL_2 action on $\Omega'(\lambda_\bullet)$ commutes with the projection onto $(\mathbb{P}^1)^k$ and furthermore it preserves $(\mathbb{P}^1)^k \backslash \Delta$.

Definition 4.1.9. The quotient of $\Omega'(\lambda_\bullet)|_{(\mathbb{P}^1)^k-\Delta}$ by PGL_2 defines a family

$$\vartheta^\circ_{\lambda_\bullet} : \Omega(\lambda_\bullet) \longrightarrow M_{0,k}(\mathbb{C}),$$

where $\Omega(\lambda_\bullet)$ is the quotient $\Omega'(\lambda_\bullet)|_{(\mathbb{P}^1)^k-\Delta}/\,\mathrm{PGL}_2$.

Since PGL_2 acts by isomorphisms on the fibres of $\mathrm{Gr}(r,d) \times (\mathbb{P}^1)^1$, the fibre of $\vartheta^\circ_{\lambda_\bullet}$ over $z \in M_{0,k}(\mathbb{C})$ is isomorphic to $\Omega(\lambda_\bullet; z')$, where z' is any representative of z.

4.1.3 Littlewood Richardson coefficients

Recall if $\lambda \in \mathtt{Part}(r)$ then $L(\lambda)$ is the irreducible $\mathfrak{gl}_r$-representation of highest weight λ. A central aim of this thesis is to add extra structure to the following coincidence of numbers.

Theorem 4.1.10. *Let $\lambda, \mu, \nu \in \mathtt{Part}(r,d)$. The following sets all have equal cardinality, denoted $c^\nu_{\lambda\mu}$ and do not depend on r.*

(i) The set of semistandard tableaux of shape $\nu\backslash\mu$ with rectification equal to a fixed tableaux S of shape λ.

(ii) The multiplicity of the irreducible module $L(\nu)$ in $L(\lambda) \otimes L(\mu)$.

(iii) Any basis of $[L(\lambda) \otimes L(\mu)]^{\mathsf{sing}}_\nu$.

(iv) The set $[B(\lambda) \otimes B(\mu)]^{\mathsf{sing}}_\nu$.

(v) The set $\Omega(\lambda; \mathcal{E}) \cap \Omega(\mu; \mathcal{F}) \cap \Omega(\nu^{\mathsf{c}}; \mathcal{G}) \subset \mathrm{Gr}(r,d)$ for a generic choice of flags $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$.

The numbers $c^\nu_{\lambda\mu}$ are called the *Littlewood-Richardson coefficients*. A beautiful exposition of these coincidences is given in the book [Ful97]. The proof works by interpreting all these numbers in terms of symmetric polynomials. Let s_λ be the Schur polynomial corresponding to the partition λ (see [Ful97, Notation]). This is a symmetric function and the set $\{s_\lambda \,|\, \lambda \in \mathtt{Part}\}$ forms a basis for the space of symmetric functions. Expanding products out into this basis produces

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu.$$

In particular this allows us to define, for $\mu \in \texttt{Part}$ and $\lambda_\bullet$ a sequence of partitions, the numbers

$$c^{\mu}_{\lambda_\bullet} = \sum_{\nu} c^{\mu}_{\lambda_1\nu} c^{\nu}_{\lambda_2\lambda_3\cdots\lambda_n},$$

inductively. This is the

(i) coefficient of s_μ in $s_{\lambda_1}s_{\lambda_2}\cdots s_{\lambda_n}$,

(ii) the multiplicity of $L(\mu)$ in $L(\lambda_\bullet)$,

(iii) dimension of $L(\lambda_\bullet)^{\mathsf{sing}}_{\mu}$,

(iv) cardinality of $\mathtt{B}(\lambda_\bullet)^{\mathsf{sing}}_{\mu}$,

(v) cardinality of $\Omega(\mu^{\mathsf{c}};\mathcal{F}^{(0)}) \cap \Omega(\lambda_1;\mathcal{F}^{(1)}) \cap \ldots \cap \Omega(\lambda_n;\mathcal{F}^{(n)})$ for a generic choice of flags.

4.2 Speyer's compactification

The aim of this section is to present the construction of Speyer's compactification of $\Omega(\lambda_\bullet)$, which is done by extending this to a family over $\overline{M}_{0,k}(\mathbb{C})$. Fix the following data:

- positive integers d, r and k such that $r \leq d$, and
- a sequence of partitions $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_i \in \texttt{Part}(r,d)$ and $k \leq r(d-r)$.

Often we will reduce to the case when $\lambda_\bullet = (\square^k)$, i.e. $\lambda_i = \square$ for all $i \in [k]$. We will refer to this as the *fundamental case.*

4.2.1 The construction

If A is a three element subset of $[k]$, fix a curve C_A isomorphic to $\mathbb{P}^1$ with three points marked by the elements of A. Since A consists of exactly three elements, the choice of C_A is unique up to projective equivalence. Write $\mathrm{Gr}(r,d)_A$ for $\mathrm{Gr}(r,d)_{C_A}$. For a curve $C \in M_{0,k}(\mathbb{C})$ with marked points $z = (z_1, z_2, \ldots, z_k)$ let $\phi_A(C)\colon \mathbb{P}^1 \longrightarrow C_A$ be the unique isomorphism that, for each $a \in A$, sends $z_a \in \mathbb{P}^1$ to the point on C_A marked by a. This defines a morphism

$$\phi_A\colon M_{0,k}(\mathbb{C}) \times \mathbb{P}^1 \longrightarrow M_{0,k}(\mathbb{C}) \times C_A.$$

Applying the Grassmannian construction to this family of curves results in a morphism

$$\phi_A\colon M_{0,k}(\mathbb{C}) \times \mathrm{Gr}(r,d)_{\mathbb{P}^1} \longrightarrow M_{0,k}(\mathbb{C}) \times \mathrm{Gr}(r,d)_A.$$

The construction of Speyer's compactification takes place in a large product of Grassmanians $\overline{M}_{0,k}(\mathbb{C}) \times \prod_A \mathrm{Gr}(r,d)_A$ where A ranges over all three element subsets of $[k]$. Using the morphisms ϕ_A construct an embedding into a product of Grassmanians

$$\begin{aligned} M_{0,k}(\mathbb{C}) \times \mathrm{Gr}(r,d) &\hookrightarrow \overline{M}_{0,k}(\mathbb{C}) \times \textstyle\prod_A \mathrm{Gr}(r,d)_A \\ (C,X) &\longmapsto (C, \phi_A(C,X)). \end{aligned}$$

Identify $M_{0,k}(\mathbb{C}) \times \mathrm{Gr}(r,d)$ with its image in $\overline{M}_{0,k}(\mathbb{C}) \times \prod_A \mathrm{Gr}(r,d)_A$. We will abuse notation and use $\Omega(\lambda; z)_B$ to also denote its preimage under the projection

$$\overline{M}_{0,k}(\mathbb{C}) \times \prod_A \mathrm{Gr}(r,d)_A \to \mathrm{Gr}(r,d)_B,$$

the context should make it clear when we are referring to this preimage.

Definition 4.2.1 ([Spe14, Section 3]). The family $\mathcal{G}(r,d)$ is defined as the closure of $M_{0,k}(\mathbb{C}) \times \mathrm{Gr}(r,d)$ in $\overline{M}_{0,k}(\mathbb{C}) \times \prod_A \mathrm{Gr}(r,d)_A$. We also define the subvariety $\mathcal{S}(\lambda_\bullet)$ as $\mathcal{G}(r,d) \cap \bigcap_{a \in A} \Omega(\lambda_a, a)_A$, where the intersection ranges over all pairs (a, A) for A a three element subset of $[k]$ and $a \in A$.

Theorem 4.2.2 ([Spe14, Theorem 1.1]). *The family $\mathcal{G}(r,d)$ and its subfamily $\mathcal{S}(\lambda_\bullet)$ have the following properties:*

(i) $\mathcal{G}(r,d)$ and $\mathcal{S}(\lambda_\bullet)$ are Cohen-Macaulay and flat over $\overline{M}_{0,k}(\mathbb{C})$.

(ii) $\mathcal{G}(r,d)$ is isomorphic to $\mathrm{Gr}(r,d) \times M_{0,k}(\mathbb{C})$ over $M_{0,k}(\mathbb{C})$.

(iii) $\mathcal{S}(\lambda_\bullet)$ is isomorphic to $\Omega(\lambda_\bullet)$ over $M_{0,k}(\mathbb{C})$.

(iv) If a representative of $C \in M_{0,k}(\mathbb{C})$ is given by $\mathbb{P}^1$ with the marked points at $z = (z_1, z_2, \ldots z_k) \in (\mathbb{P}^1)^k$ then the fibre of $\mathcal{S}(\lambda_\bullet)$ over the point C is isomorphic to $\Omega(\lambda_\bullet; z)$.

Theorem 4.2.3 ([Spe14, Theorem 1.4]). *If $|\lambda_\bullet| = \sum |\lambda_i| = r(d-r)$, the fibre of $\mathcal{S}(\lambda_\bullet)$ over $C \in \overline{M}_{0,k}(\mathbb{R})$ is a reduced union of real points and $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$ is a topological covering space over $\overline{M}_{0,k}(\mathbb{R})$.*

4.2.2 The fibre

We will also want an explicit description of the fibres of $\mathcal{S}(\lambda_\bullet)$ so let us recall this from [Spe14]. Fix $C \in \overline{M}_{0,k}(\mathbb{C})$, a not necessarily irreducible curve and denote its irreducible components $C_1, C_2, \ldots C_l$. Fix an irreducible component C_i and let $A \subseteq [k]$ be a three element subset. If $d_1, \ldots, d_e$ are the nodes lying on C_i we say that $v(A) = C_i$ if the points marked by A lie on three separate connected components of $C \setminus \{d_1, \ldots, d_e\}$.

Define the *projection of* $a \in [n]$ onto C_i: if a marks a point on C_i then the projection is a, otherwise there is a unique node $d \in C_i$ via which a is path connected to C_i, let d be the projection. If $v(A) = C_i$ then the projection of A onto C_i produces three distinct points on C_i.

If $v(A) = C_i$ define the isomorphism $\phi_{i,A} : C_i \longrightarrow C_A$ given sending the projection onto C_i of $a \in A$ to the point marked by a in C_A. This morphism is uniquely determined since $v(A) = C_i$. By considering the corresponding isomorphisms $\mathrm{Gr}(r,d)_{C_i} \longrightarrow \mathrm{Gr}(r,d)_A$ we obtain an embedding

$$\mathrm{Gr}(r,d)_{C_i} \hookrightarrow \textstyle\prod_{v(A)=C_i} \mathrm{Gr}(r,d)_A.$$

Identify $\mathrm{Gr}(r,d)_{C_i}$ with its image. Speyer shows the projection from $\mathcal{G}(r,d)$ into $\prod_{v(A)=C_i} \mathrm{Gr}(r,d)_A$ lands inside $\mathrm{Gr}(r,d)_{C_i}$. In this way one may think of the fibre of $\mathcal{G}(r,d)$ over C as a subvariety of $\prod_i \mathrm{Gr}(r,d)_{C_i}$. We again make the abuse of notation and identify $\Omega(\lambda; z)_{C_i}$ with its preimage under the projection

$$\overline{M}_{0,k}(\mathbb{C}) \times \prod_A \mathrm{Gr}(r,d)_A \to \prod_{v(A)=C_i} \mathrm{Gr}(r,d)_A \supset \mathrm{Gr}(r,d)_{C_i}.$$

Definition 4.2.4. A *node labelling* for C is a function ν which assigns to every pair (C_i, d), of an irreducible component and node $d \in C_j$, a partition $\nu(C_j, x) \in \mathtt{Part}(r,d)$ such that if $d \in C_i \cap C_j$ then $\nu(C_i,d)^{\mathsf{c}} = \nu(C_j,d)$. Denote the set of all node labellings by $\mathtt{N}_C$.

Theorem 4.2.5 ([Spe14, Section 3, proof of Theorem 1.2])**.** *Let $C \in \overline{M}_{0,k}(\mathbb{C})$ be a stable curve with irreducible components $C_1, C_2, \ldots, C_l$. Let D_i be the set of nodes on C_i and P_i the set of marked points. The fibres of $\mathcal{G}(r,d)$ and $\mathcal{S}(\lambda_\bullet)$ over C are*

$$\mathcal{G}(r,d)(C) = \bigcup_{\nu \in \mathtt{N}_C} \prod_i \bigcap_{d \in D_i} \Omega(\nu(C_i,d); d)_{C_i}, \tag{4.2.1}$$

$$\mathcal{S}(\lambda_\bullet)(C) = \bigcup_{\nu \in \mathtt{N}_C} \prod_i \left(\bigcap_{d \in D_i} \Omega(\nu(C_i,d), d)_{C_i} \cap \bigcap_{p \in P_i} \Omega(\lambda_p, p)_{C_i} \right). \tag{4.2.2}$$

4.3 Growth diagrams

The real points of Speyer's compactification of $\Omega(\lambda_\bullet)$ admit a very explicit combinatorial description. In order to give this description we first recall some combinatorial notions and record some of their fundamental properties. In particular we will need an interpretation of jeu de taquin slides for standard tableaux using combinatorial objects built on subsets of the lattice $\mathbb{Z}^2$. When drawing this lattice, the second coordinate will be depicted increasing northward on the vertical axis and (perhaps counter intuitively) the first coordinate increasing *westward* on the horizontal axis. This is shown in Figure 4.2. This

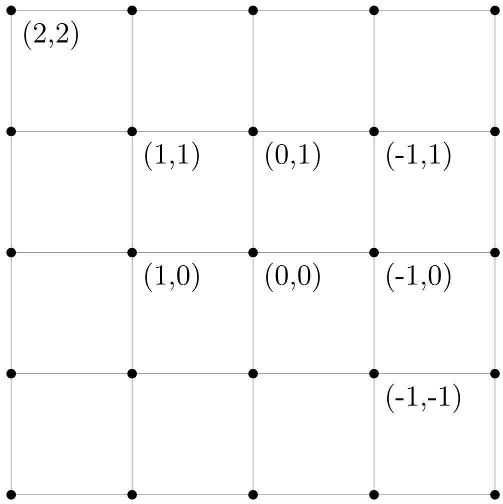


Figure 4.2: Convention for depicting $\mathbb{Z}^2$

means the southeast to northwest diagonal passing through the point $(0,k)$ will consist of exactly the pairs (i,j) such that $j-i=k$. This choice is made in order to be consistent with the notation in [Spe14].

4.3.1 Definition of Growth diagrams

Define the subset $\mathbb{Z}^2_+ \subset \mathbb{Z}^2$ defined by

$$\mathbb{Z}^2_+ = \left\{ (i,j) \in \mathbb{Z}^2 \,\middle|\, j-i \geq 0 \right\}.$$

Definition 4.3.1. If $\mathbb{I} \subset \mathbb{Z}^2_+$, a *growth diagram* on $\mathbb{I}$ is a map $\gamma : \mathbb{I} \longrightarrow \texttt{Part}$ obeying the following rules:

(i) If $j-i=k\geq 0$ then γ_{ij} is a partition of k.

(ii) Suppose $(i,j) \in \mathbb{I}$. Then if $(i-1,j)$ (respectively $(i,j+1)$) is in $\mathbb{I}$ then $\gamma_{ij} \subset \gamma_{(i-1)j}$ (respectively $\gamma_{ij} \subset \gamma_{i(j+1)}$).

(iii) If $(i,j),(i-1,j),(i,j+1)$ and $(i-1,j+1) \in \mathbb{I}$ and $\gamma_{(i-1)(j+1)} \backslash \gamma_{ij}$ consists of two boxes that *do not* share an edge then $\gamma_{(i-1)j} \neq \gamma_{i(j+1)}$.

In view of condition (i), condition (ii) means that moving one step north or one step east in $\mathbb{I}$, adds a single box. Condition (iii) means that given an entire square in $\mathbb{I}$, and if there are two possible ways to go from γ_{ij} to $\gamma_{(i-1)(j+1)}$ by adding boxes then the two paths around the square should be these two different ways. Take the region $\mathbb{I} = \{(i,j) \,|\, i=0,-1,-2 \text{ and } j=0,1\}$, then an example of a growth diagram for $\mathbb{I}$ is given in Figure 4.3.

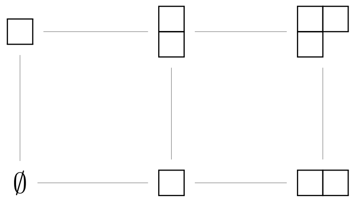


Figure 4.3: An example of a growth diagram

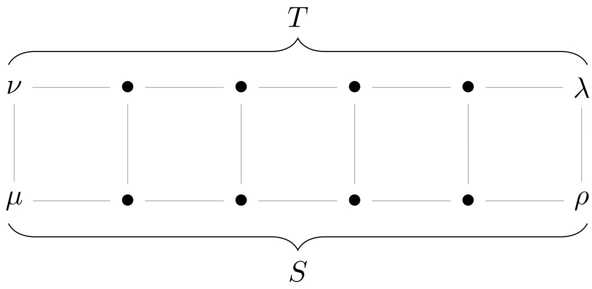


Figure 4.4: The jeu de taquin growth diagram

A *path* through $\mathbb{I} \subset \mathbb{Z}_+^2$ is a connected series of steps from one vertex to another using only northward and eastward moves (i.e. only ever increasing j and decreasing i so that $j-i$ is a strictly increasing function on the path). Given a growth diagram γ in $\mathbb{I}$, every path determines a standard tableau. For example the sequence $(0,0)-(-1,0)-(-1,1)-(-2,1)$ in the growth diagram given in Figure 4.3 is a path and determines the standard tableau

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}.$$

Given a rectangular region in a growth diagram, conditions (i) and (iii) mean the entire region is determined by specifying the tableaux along any path from its bottom left corner to the top right.

4.3.2 Jeu de taquin in growth diagrams

Now we return to the claim that growth diagrams encode the jeu de taquin slides on standard tableaux. Let $\mathbb{I}$ be a rectangular region in $\mathbb{Z}_+^2$ only one step tall. Say $\mathbb{I} = \{(i,j)\,|\, i = r, r+1\ldots, s \text{ and } j = t, t+1\}$, so that (s,t) is the bottom left hand corner and $t-s \geq 0$. Let γ be a growth diagram on $\mathbb{I}$ and set $\mu = \gamma_{st}, \nu = \gamma_{s(t+1)}, \rho = \gamma_{rt}$ and $\lambda = \gamma_{r(t+1)}$. These are the four partitions at the corners of $\mathbb{I}$.

Let T be the $\lambda\backslash\nu$-tableau given by the top edge of $\mathbb{I}$ and S the $\rho\backslash\mu$-tableau given by the bottom edge. See Figure 4.4. The partition μ determines an addable inside node (see Definition 3.2.5) for T denoted by $\circ$ and similarly the partition λ determines an addable outside node for S denoted $*$.

Proposition 4.3.2. *The tableau S is the result of the reverse slide of T into $\circ$, and T is the result of the forward slide of S into $*$.*

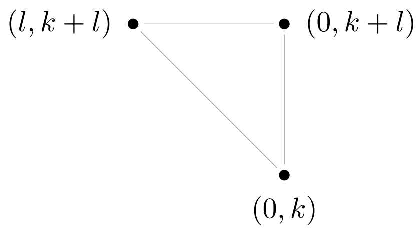
Proof. See [Sta99], Proposition A1.2.7. □

4.3.3 Schützenberger involution in growth diagrams

We now explain how this relates to the Schützenberger involution for standard tableaux. Suppose $T \in \mathtt{SYT}(\lambda \backslash \mu)$ where μ is a partition of k and $\lambda\backslash\mu$ has l boxes. Let

$$\mathbb{I} = \left\{ (i,j) \in \mathbb{Z}_+^2 \,\middle|\, 0 \leq i \leq l, k \leq j \leq k+l, \text{ and } i+k \leq j \right\}.$$

That is, $\mathbb{I}$ is a triangle with vertices $(0,k),(l,k+l)$ and $(0,k+l)$ as depicted below.



Note that given a triangular region $\mathbb{I}$ as above, condition iii in Definition 4.3.1 means that any growth diagram γ on $\mathbb{I}$ can be computed recursively if we know γ either of the horizontal or vertical sides of $\mathbb{I}$.

Define the growth diagram γ_T on $\mathbb{I}$ by setting $\gamma_T(r, k+r) = \mu$ for any $0 \leq r \leq l$. That is, on the diagonal edge of $\mathbb{I}$, γ_T is of constant value μ. We set the sequence of partitions

$$\gamma_T(l, k+l) \subset \gamma_T(l-1, k+l) \subset \ldots \subset \gamma_T(0, k+l)$$

on the horizontal edge of $\mathbb{I}$ so they determine the standard tableau T (see Remark 3.2.3). By the observation above, this determines γ_T on all of $\mathbb{I}$.

Definition 4.3.3. The *Schützenberger growth diagram* of $T \in \mathtt{SYT}(\lambda\backslash\mu)$ is γ_T.

An immediate consequence of the definition and of Proposition 4.3.2 is the following corollary. This explains why the Schützenberger involution is in fact an involution.

Corollary 4.3.4. *Let γ_T be the Schützenberger growth diagram of a standard tableau $T \in \mathtt{SYT}(\lambda\backslash\mu)$. The standard tableau determined by the sequence of partitions*

$$\gamma_T(0, k) \subset \gamma_T(0, k+1) \subset \ldots \subset \gamma_T(0, k+l)$$

along the vertical edge of $\mathbb{I}$ is $\mathtt{evac}(T)$, the Schützenberger involution of T.

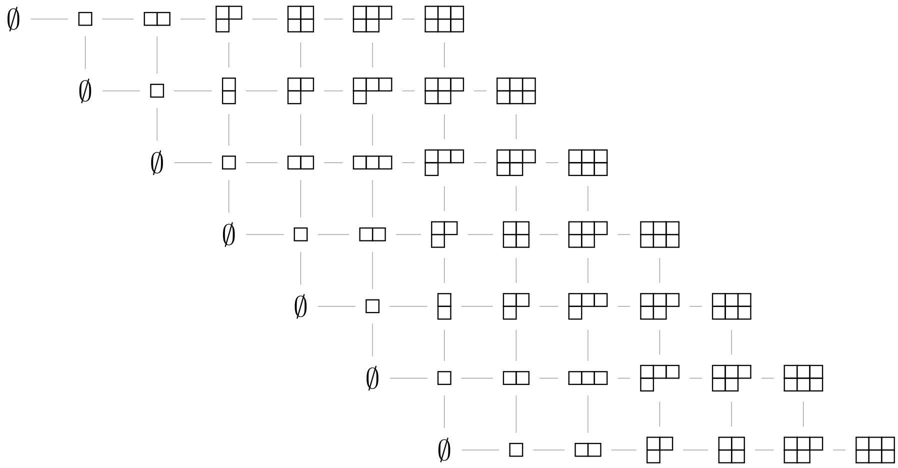


Figure 4.5: An example of a growth diagram for $r = 2$ and $d = 5$. We can take the bottom left corner as $(1, 1)$.

4.3.4 Cylindrical growth diagrams

For this section, fix positive integers $r \leq d$ and let $k = r(d - r)$. Recall that we denote the partition with r rows and $d-r$ columns by $\Lambda_{r,d}$. Consider the subset $\mathbb{I} \subset \mathbb{Z}_+^2$ defined by

$$\mathbb{I}_k = \left\{ (i,j) \in \mathbb{Z}_+^2 \,\middle|\, j - i \leq k \right\}.$$

Definition 4.3.5. A growth diagram on $\mathbb{I}_k$ will be called a *cylindrical growth diagram* for (r, d) if it satisfies the condition that $\gamma_{i(i+k)}$ is $\Lambda_{r,d}$.

The reason for the adjective cylindrical is that these growth diagrams will turn out to be periodic along the north-west diagonal. An example of (part of) a cylindrical growth diagram for $r = 2$ and $d = 5$ is given in Figure 4.5. As one can see from the diagram, the bottom row is repeated at the top. This is in fact a general phenomenon and is why these diagrams are given the adjective cylindrical.

Remark 4.3.6. A path through γ, as defined in Section 4.3.1, from a node (i, i) to a node $(j, j + k)$ (i.e. nodes lying on the left and right edges of the diagram) completely defines all of the partitions lying in the rectangular region the path spans. In the case of cylindrical diagrams the extra condition that $\gamma_{i(i+k)} = \Lambda_{r,d}$ means such a path determines the cylindrical growth diagram completely.

4.4 Dual equivalence

We now describe Haiman's notion of dual equivalence. This is an equivalence relation on tableaux dual to slide equivalence in the sense that it preserves the Q-symbol of a word (recall that slide equivalence preserves the P-symbol of a word). Let $\mu \subset \lambda \subset \nu$ be partitions and let T be a semistandard $\lambda\backslash\mu$-tableau, then any standard tableau X of shape $\nu\backslash\lambda$ (we say X has shape *extending* T) defines a sequence of slides we can apply to T (i.e. first slide into the box numbered 1, then into the box numbered 2 and so on). Denote the resulting tableaux $\mathrm{jdt}_X(T)$.

Definition 4.4.1. Two semistandard tableaux, T and T' of the same shape, are called *dual equivalent*, denoted $T \sim_D T'$, if for any standard tableau X of shape extending T and T', the tableaux $\mathrm{jdt}_X(T)$ and $\mathrm{jdt}_X(T')$ have the same shape.

Given two tableaux, it is difficult to show they are dual equivalent directly. Below we will recall some theorems which we will use to check dual equivalence in practice. A trivial example of dual equivalent tableaux are two (semistandard) tableaux with only a single box, however these boxes are slid around they will still remain only a single box and thus the same shape. Two tableaux which are *not* dual equivalent are

$$\begin{array}{ccc} & & \boxed{3} \\ & \boxed{2} & \\ \boxed{1} & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & \boxed{2} \\ & \boxed{3} & \\ \boxed{1} & & \end{array}.$$

For example, the forward slide into the position between the 2 and 3 results in tableaux of different shape.

4.4.1 Dual equivalence is local

We have the following proposition which tells us that dual equivalence is a *local* operation. That is, we can replace a subtableau with a dual equivalent one and the resulting tableau will be dual equivalent to the original one.

Proposition 4.4.2 (Lemma 2.1 in [Hai92])**.** *Suppose X, Y, S and T are semistandard tableaux such that $X \cup T \cup Y$ and $X \cup S \cup Y$ are semistandard tableaux. If $S \sim_D T$ then $X \cup T \cup Y \sim_D X \cup S \cup Y$.*

A skew-shape is called *antinormal* if it has a unique bottom right corner. That is, if it is the south-eastern part of a rectangle. We have the following important properties of dual equivalence which we will use later.

Theorem 4.4.3 ([Hai92])**.** *Dual equivalence has the following properties.*

(i) All tableaux of a given normal or antinormal shape are dual equivalent.

(ii) The intersection of any slide equivalence class and any dual equivalence class is a unique tableaux.

(iii) Two words are dual equivalent if and only if their Q-symbols agree.

Proof. Properties i, ii and iii are Proposition 2.14, Theorem 2.13 and Theorem 2.12 in [Hai92] respectively. $\square$

4.4.2 Shuffling dual equivalence classes

Given a rectangular growth diagram let S_1 and S_2 be the standard tableaux defined by the western edge and the northern edge respectively and let T_1 and T_2 denote the standard tableaux defined by the southern and eastern edges respectively.

Proposition 4.4.4 (Proposition 7.6 in [Spe14])**.** *The dual equivalence classes of T_1 and T_2 remain unchanged if we replace either (or both) S_1 or S_2 by dual equivalent tableaux.*

Let δ_1 and δ_2 be dual equivalence classes such that the shape of δ_2 extends the shape of δ_1. Choose representatives S_1 and S_2 for δ_1 and δ_2 respectively. Construct the unique rectangular growth diagram with western and northern edges given by S_1 and S_2 respectively. Define $\varepsilon_1(\delta_1, \delta_2)$ and $\varepsilon_2(\delta_1, \delta_2)$ to be the dual equivalence classes of the southern and eastern edges respectively. Proposition 4.4.4 implies that ε_1 and ε_2 are independent of the representatives S_1 and S_2 chosen.

Definition 4.4.5. If δ_1 and δ_2 are dual equivalence classes such that the shape of δ_2 extends the shape of δ_1 we say $\varepsilon_1(\delta_1, \delta_2)$ and $\varepsilon_2(\delta_1, \delta_2)$ are the dual equivalence classes given by *shuffling* δ_1 and δ_2.

As an example take δ_1 to be the unique dual equivalent class in the set of standard tableaux of shape $\square$ and δ_2 to be the dual equivalence class of the tableau

$$\begin{array}{cc} & \boxed{2} \\ \boxed{1} & \end{array}.$$

Following the procedure described above we obtain the growth diagram from Figure 4.3. Hence $\varepsilon_1(\delta_1, \delta_2)$ is the unique dual equivalence class of shape (2) and $\varepsilon_2(\delta_1, \delta_2)$ is the unique dual equivalence class of shape $(2, 1)\backslash(1, 1)$.

4.4.3 Dual equivalence growth diagrams

We now introduce a notion of dual equivalence for growth diagrams. These are more general notions of growth diagrams in subsets $\mathbb{Z}_+^2$ in which we allow the partitions to grow by more than a single box each step. First fix a function $m : \mathbb{Z} \longrightarrow \mathbb{Z}_{>0}$ called the *interval*, this controls how many boxes we are allowed to add at each step. We then define several auxiliary functions using m; a function $\hat{m} : \mathbb{Z} \longrightarrow \mathbb{Z}$

$$\hat{m}(i) = \begin{cases} 1 + \sum_{k=1}^{i-1} m(k) & \text{if } i > 0 \\ 1 - \sum_{k=i}^{0} m(k) & \text{if } i \leq 0, \end{cases}$$

a function $\bar{m} : \mathbb{Z}_+^2 \longrightarrow \mathbb{Z}_+^2$

$$\bar{m}(i,j) = (\hat{m}(i), \hat{m}(j)),$$

and a function $m_s : \mathbb{Z}_+^2 \longrightarrow \mathbb{Z}_{\geq 0}$

$$m_s(i,j) = \hat{m}(j) - \hat{m}(i) = \sum_{k=i}^{j-1} m(k).$$

In particular $m_s(i,i) = 0$, $m_s(i,i+1) = m(i)$ and if m is simply the constant function 1 then $m_s(i,j) = j - i$.

Consider the graph with vertices $\mathbb{Z}_+^2$ and edges a_{ij} between (i,j) and $(i-1,j)$ and edges b_{ij} between vertices (i,j) and $(i,j+1)$. If we embed $\mathbb{Z}_+^2 \subset \mathbb{R}^2$, then we can think of these as simply the horizontal and vertical unit intervals between the points of $\mathbb{Z}_+^2$. If $\mathbb{I} \subset \mathbb{Z}_+^2$, we call an edge of $\mathbb{Z}_+^2$ *internal* to $\mathbb{I}$ if both of its endpoints are in $\mathbb{I}$.

Definition 4.4.6. A *dual equivalence growth diagram* in $\mathbb{I}$ with interval m is a map $\gamma : \mathbb{I} \longrightarrow \texttt{Part}$ as well as an assignment of a dual equivalence class α_{ij} (respectively β_{ij}) to every edge a_{ij} (respectively b_{ij}) internal to $\mathbb{I}$, obeying the following rules.

(i) If $(i,j) \in \mathbb{I}$ then γ_{ij} is a partition of $m_s(i,j)$.

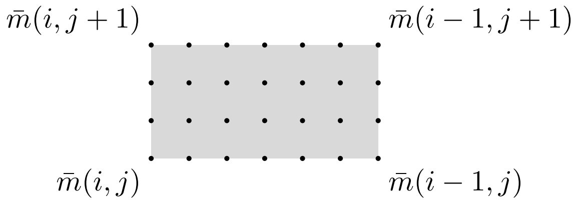
(ii) If a_{ij} (respectively b_{ij}) is internal to $\mathbb{I}$ then $\gamma_{ij} \subset \gamma_{(i-1)j}$ (respectively $\gamma_{ij} \subset \gamma_{i(j+1)}$).

(iii) If $a_{ij}, b_{(i-1)j}, b_{ij}$ and $a_{i(j+1)}$ are all elements of $\mathbb{I}$ then $\varepsilon_1(\beta_{ij}, \alpha_{i(j+1)}) = \alpha_{ij}$ and $\varepsilon_2(\beta_{ij}, \alpha_{i(j+1)}) = \beta_{(i-1)j}$.

As stated above, the interval m defines how many boxes we are allowed to add with each step though the lattice. Indeed, if we wish to move one step east from (i,j), m_s increases by $m(i-1)$ and if we wish to move one step north, m_s increases by $m(j)$. Thus the partition γ_{ii} is the empty partition and $\gamma_{i(i+1)}$ is a partition of $m(i)$.

This means when m is the constant function 1 the definition coincides with that for a ordinary growth diagram for $\mathbb{I}$. This is clear since in this case $m_s(i,j) = j - i$, condition ii remains unchanged and since we are only adding a single box with each step there is only a single dual equivalence class.

Certain classes of dual equivalence growth diagrams will be the central combinatorial objects which we study. We can also think of dual equivalence growth diagrams as equivalence classes of certain growth diagrams. Let $\tilde{\gamma}$ be a growth diagram on $\tilde{\mathbb{I}} \subset \mathbb{Z}_+^2$. We say $\tilde{\mathbb{I}}$ is *adapted* to an interval $m : \mathbb{Z} \longrightarrow \mathbb{Z}_{>0}$ if it has the following property: If $\tilde{\mathbb{I}}$ contains each of the four vertices



then $\tilde{\mathbb{I}}$ contains each vertex inside (and on the boundary of) the rectangular they region bound.

Definition 4.4.7. If $\tilde{\mathbb{I}}$ is adapted to m, the *reduction modulo m* of a growth diagram $\tilde{\gamma}$ is defined to be the map $\gamma : \mathbb{I} \longrightarrow \texttt{Part}$ for

$$\mathbb{I} = \left\{ (i, j) \in \mathbb{Z}_+^2 \,\middle|\, (\hat{m}(i), \hat{m}(j)) \in \tilde{\mathbb{I}} \right\},$$

given by $\gamma = \tilde{\gamma} \circ \bar{m}$, along with the set of dual equivalence classes

- α_{ij}, the dual equivalence class defined by the horizontal path from $\bar{m}(i, j)$ to $\bar{m}(i-1, j)$ and
- β_{ij}, the dual equivalence class of the tableaux defined by the vertical path from $\bar{m}(i, j)$ to $\bar{m}(i-1, j)$.

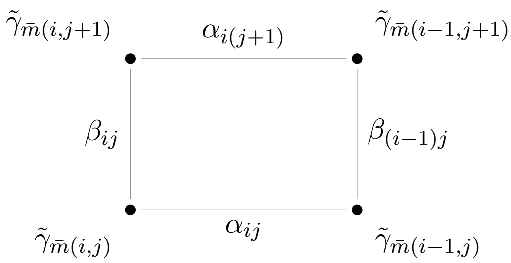
Proposition 4.4.8. *The map $\gamma : \mathbb{I} \longrightarrow \texttt{Part}$ along with the choice of α_{ij} and β_{ij} define a dual equivalence growth diagram on $\mathbb{I}$.*

Proof. We must check the conditions in Definition 4.4.6. For condition i note that $\gamma_{ij} = \tilde{\gamma}_{\hat{m}(i), \hat{m}(j)}$ so $|\gamma_{ij}| = \hat{m}(j) - \hat{m}(i)$ which is $m_s(i, j)$ by definition. We have a path in $\tilde{\mathbb{I}}$ from $\bar{m}(i, j)$ to $\bar{m}(i-1, j)$ so

$$\gamma_{ij} = \tilde{\gamma}_{\bar{m}(i,j)} \subset \tilde{\gamma}_{\bar{m}(i-1,j)} = \gamma_{(i-1)j}.$$

Similarly $\gamma_{ij} \subset \gamma_{i(j+1)}$.

To see condition iii, note that $\alpha_{ij}, \beta_{(i-1)j}, \beta_{ij}$ and $\alpha_{i(j+1)}$ are defined as the dual equivalence classes coming from the four sides of a rectangular growth diagram:



The fact that this portion of $\tilde{\mathbb{I}}$ forms a rectangular growth diagram is given by the requirement that $\tilde{\mathbb{I}}$ is adapted to m. By definition, this means $\beta_{ij}, \alpha_{i(j+1)}$ is the shuffle of $\alpha_{ij}, \beta_{(i-1)j}$ as required. $\square$

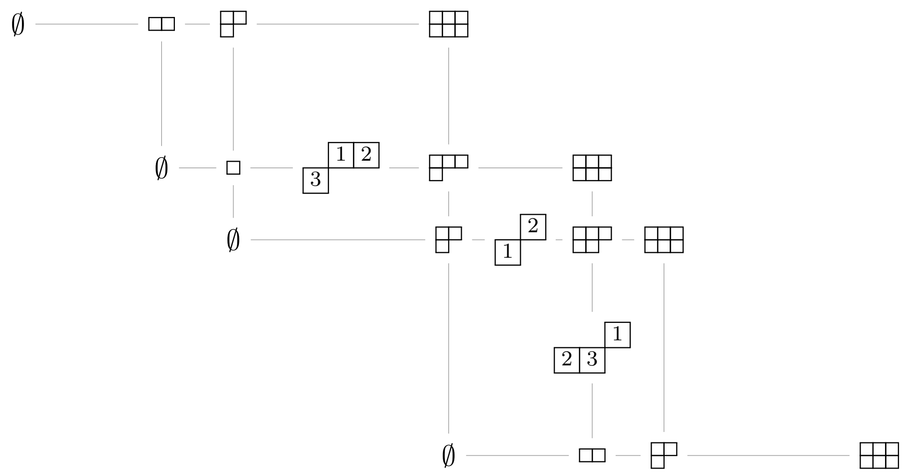


Figure 4.6: An example of a dual equivalence cylindrical growth diagram for $r = 5$ and $d = 2$ and interval $m(i) = a_{(i \mod 3)}$ where $(a_1, a_2, a_3) = (3, 1, 2)$.

4.4.4 Dual equivalence cylindical growth diagrams

We now recall Speyer's dual equivalence classes of cylindrical growth diagrams. As before fix $r \leq d$. However now choose $k \leq r(d-r)$ and a sequence of partitions $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

$$|\lambda_\bullet| = \sum_{i=1}^{k} |\lambda_i| = r(d-r).$$

It will be convenient to always take indices, when referring to this sequence, modulo k. That is, λ_l will always mean $\lambda_{(l \mod k)}$. With this convention in mind set $m(i) = |\lambda_i|$.

Definition 4.4.9. A *dual equivalence cylindrical growth diagram* (or decgd for short) of *shape* $\lambda_\bullet$ is a dual equivalence growth diagram γ on $\mathbb{I}_k$ with $\gamma_{i(i+k)} = \Lambda_{r,d}$, and such that $\gamma_{i(i+1)} = \lambda_i$. We denote the set of decgd's of shape $\lambda_\bullet$ by $\texttt{decgd}(\lambda_\bullet)$.

As an example take $r = 2$ and $d = 5$. If we choose

$$\lambda_\bullet = \left(\begin{smallmatrix} \square\square \\ \square \end{smallmatrix}, \square, \square\square \right)$$

then Figure 4.6 gives an example of a decgd of shape $\lambda_\bullet$. Note that since shapes with only a single box, as well as shapes of normal or antinormal shape (see Theorem 4.4.3 (i)) only have a single dual equivalence class we do not indicate the dual equivalence class for edges that correspond to such a shape.

4.4.5 Reduction of cylindrical growth diagrams

The reason for the strange choice of layout in Figure 4.6 is the following. If we superimpose the diagram on top of Figure 4.5 we can see that it was simply obtained by forgetting certain nodes in Figure 4.5 but remembering the dual equivalence classes defined by the paths between nodes that we kept. In fact it is the reduction modulo m of the cylindrical growth diagram from Figure 4.5.

As above fix $r \leq d$ and a sequence of partitions $\lambda_\bullet = (\lambda_1, \lambda_2, \dots, \lambda_k)$ for $k \leq r(r-d)$ and such that $|\lambda_\bullet| = r(r-d)$, also set $\tilde{k} = r(d-r)$.

Lemma 4.4.10. *The set $\mathbb{I}_{\tilde{k}}$ is adapted to $m(i) = |\lambda_i|$ and*

$$\left\{(i,j) \in \mathbb{Z}_+^2 \,\middle|\, \bar{m}(i,j) \in \mathbb{I}_{\tilde{k}}\right\} = \mathbb{I}_k.$$

That is, the reduction modulo m of a cylindrical growth diagram on $\mathbb{I}_{\tilde{k}}$ for (r,d) is a decgd on $\mathbb{I}_k$ for (r,d) of shape

$$\gamma_{\bar{m}(1,2)}, \gamma_{\bar{m}(2,3)}, \dots, \gamma_{\bar{m}(k,k+1)}.$$

Proof. By overlapping $\mathbb{I}_{\tilde{k}}$ with any rectangular region we see that the only way for the rectangle to be contained in $\mathbb{I}_{\tilde{k}}$ completely is if one of its corners is not contained in $\mathbb{I}_{\tilde{k}}$. Hence $\mathbb{I}_{\tilde{k}}$ is adapted to m (and in fact to any interval).

Suppose that $(i,j) \in \mathbb{I}_k$, thus $j - i \leq k$. We would like to show $\bar{m}(i,j) \in \mathbb{I}_{\tilde{k}}$, that is we would like to show $\hat{m}(j) - \hat{m}(i) = m_s(i,j) \leq \tilde{k}$. But

$$m_s(i,j) = \sum_{l=i}^{j-1} m(l) = \sum_{l=i}^{j-1} \left|\lambda_{(l \mod k)}\right|,$$

and since $j - i \leq k$ each λ_l occurs at most once in the above sum. By the assumption that $|\lambda_\bullet| = \tilde{k}$ we have $m_s(i,j) \leq \tilde{k}$ as required. Now consider $(i,j) \in \mathbb{Z}_+^2$ such that $m_s(i,j) \leq \tilde{k}$. If $j - i > k$ then by the pigeonhole principle each λ_l must occur at least once, and some λ_l must occur twice in the sum

$$m_s(i,j) = \sum_{l=i}^{j-1} \left|\lambda_{(l \mod k)}\right|,$$

contradicting the fact that $m_s(i,j) \leq \tilde{k}$. This proves the second claim.

The only thing left to check, in order to ensure that the reduction modulo m of a cylindrical growth diagram on $\mathbb{I}_{\tilde{k}}$ is a decgd on $\mathbb{I}_k$, is that for any i, $\bar{m}(i, i+k) = (j, j+\tilde{k})$ for some j. Equivalently we check that $m_s(i, i+k) = \tilde{k}$. This is straightforward since the sum

$$m_s(i, i+k) = \sum_{l=i}^{i+k-1} |\lambda_l|$$

contains one of each λ_l appearing in $\lambda_\bullet$ and by assumption $|\lambda_\bullet| = \tilde{k}$. □

The key result we will need is the following.

Proposition 4.4.11 (Proposition 8.1 in [Spe14])**.** *Every decgd on $\mathbb{I}_k$ is the restriction of a cylindrical growth diagram on $\mathbb{I}_{\tilde{k}}$ and the number of decgd's of shape $\lambda_\bullet$ is $c_{\lambda_\bullet}^{\Lambda_{r,d}}$.*

4.5 Tilling of $\overline{M}_{0,k}(\mathbb{R})$ by associahedra

The combinatorial description of $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$ relies on the fact that $\overline{M}_{0,k}(\mathbb{R})$ admits a nice combinatorial description. In this section we recall a description of the CW-structure on the real points $\overline{M}_{0,k}(\mathbb{R})$. This has been investigated in [Dev99], [Kap93], and [DJS03]. Restrict the stratification given in Section 2.3.3 to $\overline{M}_{0,k}(\mathbb{R})$ to obtain subspaces

$$\overline{M}_{0,k}(\mathbb{R}) = M_1(\mathbb{R}) \supset M_2(\mathbb{R}) \supset \ldots \supset M_{k-2}(\mathbb{R}) \subset \emptyset$$

where $M_i(\mathbb{R})$ is the set of (real) stable curves with at least i irreducible components.

4.5.1 Circular orderings

Let $D_k \subset S_k$ be the dihedral group generated by $(12\ldots n)$ and the involution reversing the order of $1, 2, \ldots, n$. A circular ordering of the integers $\{1, 2, \ldots, k\}$ is an element of S_k/D_k. That is, we imagine ordering the integers on a circle and identify orderings which coincide upon rotation or reflection. The orderings $(1, 2, 3, 4)$, $(4, 1, 2, 3)$ and $(4, 3, 2, 1)$ all represent the same circular ordering but are distinct from $(1, 3, 2, 4)$.

The order in which the marked points appear on a curve $C \in M_{0,k}(\mathbb{R})$ defines a circular ordering. For each circular order $s \in S_k/D_k$, let $\Theta_s \subseteq \overline{M}_{0,k}(\mathbb{R})$ be the closure of the subspace of curves with circular ordering s. For example, Θ_{id} is the closure of the set of irreducible curves projectively equivalent to a curve with marked points at $z_1 < z_2 < \ldots < z_k$.

By a theorem of Kapranov [Kap93, Proposition 4.8], restricting the stratification to Θ_s gives it the structure of a CW-complex with i-skeleton $\Theta_s \cap M_{k-2-i}$. The symmetric group S_k acts on $\overline{M}_{0,k}(\mathbb{R})$ by permuting marked points. This action transitively permutes the cell complexes Θ_s and preserves i-cells. They are thus are all isomorphic. Define the $(k-3)$-*associahedron* to be this cell complex.

4.5.2 The fundamental group

Recall from Section 3.3.4 the cactus group, J_n, is the group with generators s_{pq} for $1 \leq p < q \leq n$ and relations

(i) $s_{pq}^2 = 1$

(ii) $s_{pq}s_{kl} = s_{kl}s_{pq}$ if the intervals $[p, q]$ and $[k, l]$ are disjoint.

(iii) $s_{pq}s_{kl} = s_{uv}s_{pq}$ if $[k, l] \subseteq [p, q]$, where $v = \hat{s}_{pq}(k)$ and $u = \hat{s}_{pq}(l)$,

where $\hat{s}_{pq}$ is the permutation that reverses the order of the interval $[p, q]$. This also provides a map to the symmetric group S_n and the pure cactus group PJ_n is defined to be the kernel of this homomorphism.

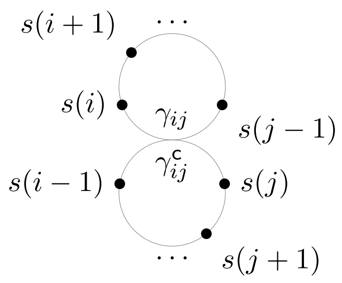


Figure 4.7: A stable curve in Θ_{ij}.

Lemma 4.5.1. *The cactus group J_n is generated by s_{1q} for $2 \leq q \leq n$.*

Proof. By the relations for J_n given above we have $s_{pq} = s_{1q}s_{1(q-p+1)}s_{1q}$. Since the elements s_{pq} generate J_n so do the s_{1q}. $\square$

In [HK06] it is shown $\pi_1(\overline{M}_{0,n+1}(\mathbb{R})) = PJ_n$. The space $\overline{M}_{0,n+1}(\mathbb{C})$ also has an action of S_n by permuting the first n marked points. This leaves the real points stable. The equivariant fundamental group $\pi_1^{S_n}(\overline{M}_{0,n+1}(\mathbb{R}))$ is J_n. See Appendix B.2 for a recollection of equivariant fundamental groups. The equivariant loop in $\overline{M}_{0,n+1}(\mathbb{R})$ corresponding to $s_{pq} \in J_n$ is $(\alpha, \hat{s}_{pq})$, where α is a simple path from a basepoint C passing through the wall reversing the labels $p, \ldots, q$ to $\hat{s}_{pq} \cdot C$.

4.6 Speyer's labelling of the fibre

We are now ready to present Speyer's combinatorial description of $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$. This will be done in two steps. First the case when $\lambda_\bullet = (\square^n)$ will be dealt with and then this case will be used to deal with the general case.

Let us restrict to the case when $|\lambda_\bullet| = r(d-r)$. By Theorem 4.2.3 the family $\mathcal{S}(\lambda_\bullet)(\mathbb{R}) \longrightarrow \overline{M}_{0,k}(\mathbb{R})$ is a topological covering of degree $c_{\lambda_\bullet}^{\Lambda_{r,d}}$. Recall from Section 4.5.1 that $\overline{M}_{0,k}(\mathbb{R})$ is tiled by associahedra. This tiling is indexed by circular orderings of the set $[k]$. We can lift the cellular structure to a tiling by associahedra of $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$ and the aim of this section will be to explain Speyer's combinatorial description of this CW-complex structure.

For now, we will just consider the fundamental case $\lambda_\bullet = (\square^k)$. Choose a circular ordering, $s = (s(1), s(2), \ldots, s(k))$, and let Θ be an associahedron of $\mathcal{S}(\square^k)(\mathbb{R})$ lying above the associahedron $\Theta_s \subset \overline{M}_{0,k}(\mathbb{R})$. The associahedron has facets labelled by non-adjacent pairs (i,j) where $i < j$. The facet Θ_{ij} of Θ lies over stable curves that generically have two components, one containing (in order) the labels $s(i), s(i+1), \ldots, s(j-1)$ and the other containing the labels $s(j), s(j+1), \ldots, s(i-1)$. Such a stable curve is depicted in Figure 4.7.

Fix a generic point C in Θ_{ij}. Theorem 4.2.5 tells us that the map ν assigns a partition to either side of the node of the stable curve at C. Let γ_{ij} be the partition assigned to the side of the node *away from* the component labelled $s(i), s(i+1), \ldots, s(j-1)$. Again see Figure 4.7 for a depiction of this situation.

Proposition 4.6.1 ([Spe14, Lemma 7.1])**.** *The partition γ_{ij} does not depend on the choice of C.*

Proposition 4.6.2 ([Spe14, Lemma 6.4 and Theorem 6.5])**.** *For an associahedron $\Theta \subset \mathcal{S}(\Box^k)(\mathbb{R})$ the map γ is a cylindrical growth diagram. The associahedra which tile $\mathcal{S}(\Box^k)(\mathbb{R})$ are labelled by pairs (s, γ) of a circular ordering s and a cylindrical growth diagram γ.*

Note the cylindrical growth diagram depends on the particular representative of the circular ordering $s \in S_k/D_k$ that we choose. If we choose another representative the cylindrical growth diagram is shifted or we take the mirror image (this comes from the action of D_k).

4.6.1 Wall crossing in the fundamental case

We now recall Speyer's description of how these associahedra are joined together. Fix an associahedron Θ in $\mathcal{S}(\Box^k)(\mathbb{R})$ labelled by (s, γ). Let $(\hat{s}, \hat{\gamma})$ be the labelling of the associahedra $\hat{\Theta}$ joined to Θ by the facet Θ_{pq}. Using the description of $\overline{M}_{0,k}(\mathbb{R})$ the circular ordering $\hat{s}$ is obtained from s by completely reversing the order of the portion of the circular ordering given by $s(p), s(p+1), \ldots, s(q-1)$.

Proposition 4.6.3 ([Spe14, Proposition 6.7])**.** *The cylindrical growth diagram $\hat{\gamma}$ is given by*

$$\hat{\gamma}_{ij} = \begin{cases} \gamma_{ij} & \text{if } [i,j] \cap [p,q] = \emptyset \text{ or } [p,q] \subseteq [i,j], \\ \gamma_{(p+q-j)(p+q-i)} & \text{if } [i,j] \subseteq [p,q]. \end{cases} \tag{4.6.1}$$

Proof. We will not repeat the proof here except to say that since the partitions γ_{ij} are constant along the relevant divisor of $\overline{M}_{0,k}(\mathbb{R})$, the partitions do not change, only the indexation relative to the circular ordering changes. Considering how the indexation changes allows one to write the conditions in (4.6.1). □

Note $\hat{\gamma}$ can be determined recursively by the information given in (4.6.1). The pairs (i, j) which appear in (4.6.1) are those for which Θ_{ij} intersects Θ_{pq}. The rule means when we cross a wall we flip certain triangles and leave others fixed. This is depicted in Figure 4.8. The red triangle is flipped about the axis shown, green triangles are fixed, and other areas are computed recursively (or using the cyclicity properties of the diagram).

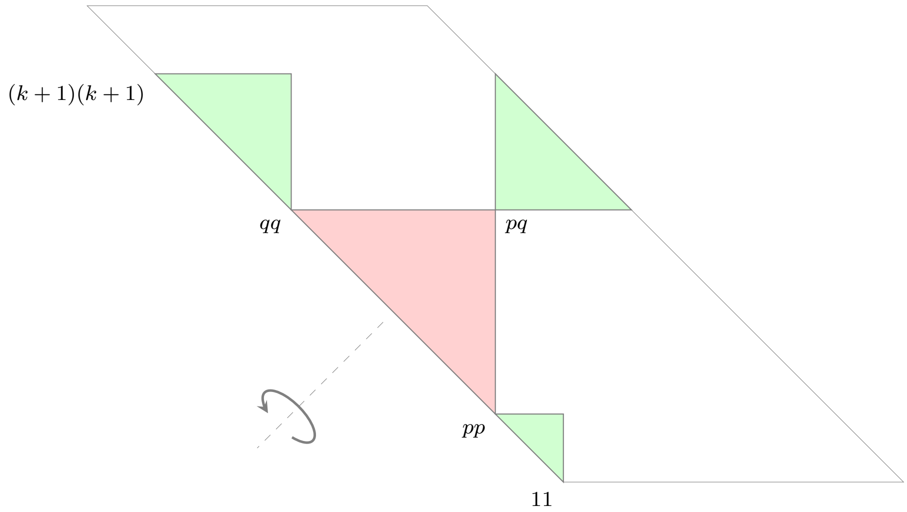


Figure 4.8: Crossing walls and flipping triangles.

4.6.2 Another realisation of $\mathcal{S}(\lambda_\bullet)$

Before we are able to describe the CW-structure for more general $\lambda_\bullet$, we realise $\mathcal{S}(\lambda_\bullet)$ as a subvariety of $\mathcal{S}(\Box^{\tilde{k}})$, in fact, the real points will be a CW-subcomplex. Here, $\tilde{k} = |\lambda_\bullet|$. We should think of obtaining the family $\mathcal{S}(\lambda_\bullet)$ inside $\mathcal{S}(\Box^{\tilde{k}})$ by colliding the first $|\lambda_1|$ marked points in such a way to obtain λ_1, the next $|\lambda_2|$ to obtain λ_2, and so on.

For our purposes here, we will take $\overline{M}_{0,2}(\mathbb{C})$ to be a single point. With this convention, there is an embedding of $\overline{M}_{0,k}(\mathbb{C}) \times \prod_{i=1}^k \overline{M}_{0,|\lambda_i|+1}(\mathbb{C})$ into $\overline{M}_{0,\tilde{k}}(\mathbb{C})$ by sending the tuple $(C, C_1, C_2, \ldots, C_k)$ to the stable curve $\tilde{C}$ obtained by the following process:

- For each i, if $|\lambda_i| \geq 2$, glue the last marked point of C_i to the i^{th} marked point of C.
- The l^{th} marked point of C_i is renumbered $l + \sum_{j=1}^{i-1} |\lambda_j|$, for $1 \leq l \leq |\lambda_i|$.
- If $|\lambda_i| = 1$ then the i^{th} marked point of C is renumbered $\sum_{j=1}^{i} |\lambda_j|$.

This is an example of a *clutching map* as described in [Knu83, Section 3] where it is also shown that this is in fact a closed embedding. Figure 4.9 shows generically what such a stable curve looks like. Restrict the family $\mathcal{S}(\Box^{\tilde{k}})$ to $\overline{M}_{0,k}(\mathbb{C}) \times \prod_{i=1}^k \overline{M}_{0,|\lambda_i|+1}(\mathbb{C})$ and let $\mathcal{Y}$ be the components where the k central nodes are labelled by $\lambda_1, \lambda_2, \ldots, \lambda_k$. The following result is used implicitly in [Spe14].

Proposition 4.6.4. *As families over $\overline{M}_{0,k}(\mathbb{C}) \times \prod_{i=1}^k \overline{M}_{0,|\lambda_i|+1}(\mathbb{C})$, the variety $\mathcal{Y}$ is isomorphic to $\mathcal{S}(\lambda_\bullet) \times \prod_{i=1}^k \mathcal{S}(\Box^{|\lambda_i|}, \lambda_i^{\mathsf{c}})$.*

Proof. First we need to set up some notation. Let $\mathbb{S}$ be the set of pairs (i, j) such that $j \in [|\lambda_i|]$, with total ordering given by the standard lexicographic ordering. There is

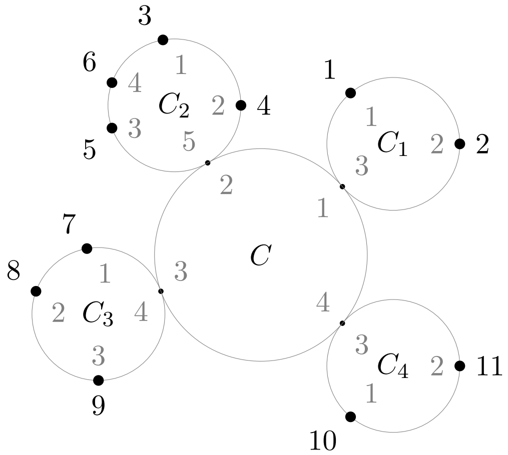


Figure 4.9: A generic point in $\overline{M}_{0,4}(\mathbb{C}) \times \prod_{i=1}^{k} \overline{M}_{0,|\lambda_i|+1}(\mathbb{C})$ when $|\lambda_2| = 4$, $|\lambda_3| = 3$ and $|\lambda_1| = |\lambda_4| = 2$. The original label of each marked point is shown in grey on the inside of each curve.

a unique order preserving bijection $\mathbb{S} \longrightarrow [\tilde{k}]$ and we will identify these two sets from now on using this bijection. Let $p\colon \mathbb{S} \longrightarrow [k]$ be the projection onto the first coordinate and given a subset $A \subset \mathbb{S}$, we denote the multiset obtained by forgetting the second coordinate by $A^\sharp$.

Let $B \subset \mathbb{S}$ be a three element subset such that $\#p(B) = 3$, denote $A = p(B)$. This condition means p is invertible when restricted to B and thus induces a bijection $A \to B$. Let $\alpha_B\colon C_A \longrightarrow C_B$ be the induced isomorphism and thus we have a corresponding isomorphism $\mathrm{Gr}(r,d)_A \to \mathrm{Gr}(r,d)_B$. We thus obtain an embedding

$$\prod_{A \subset_3 [k]} \mathrm{Gr}(r,d)_A \longrightarrow \prod_{\substack{B \subset_3 \mathbb{S} \\ \#p(B)=3}} \mathrm{Gr}(r,d)_B = \prod_{A \subset_3 [k]} \prod_{\substack{B \subset_3 \mathbb{S} \\ p(B)=A}} \mathrm{Gr}(r,d)_B$$
$$(X_A)_A \longmapsto (\alpha_B(X_{p(B)}))_B.$$

Here we use the temporary notation $\subset_3$ to mean a three element subset. Similarly, to the above there is a projection $p_a\colon \mathbb{S} \longrightarrow [|\lambda_i| + 1]$ which is defined by

$$p_a(i,j) = \begin{cases} j & \text{if } i = a, \\ \lambda_i + 1 & \text{if } i \neq a. \end{cases}$$

If $B \subset_3 \mathbb{S}$ and $\#p(B) \leq 2$, then $B^\sharp = \{i, i, *\}$ for some $i \in [k]$. Thus p_i restricts to injective function $p_i\colon B \longrightarrow [|\lambda_i| + 1]$. Let $A = p_i(B)$, as above the inverse induces an isomorphism $\alpha_{i,B}\colon C_A \longrightarrow C_B$ and we obtain an embedding for each $i \in [k]$,

$$\prod_{A\subset_3[|\lambda_i|+1]} \mathrm{Gr}(r,d)_A \longrightarrow \prod_{\substack{B\subset_3\mathbb{S}\\ \#p(B)\le 2}} \mathrm{Gr}(r,d)_B = \prod_{A\subset_3[|\lambda_i|+1]} \prod_{\substack{B\subset_3\mathbb{S}\\ p_i(B)=A}} \mathrm{Gr}(r,d)_B$$
$$(X_A)_A \longmapsto (\alpha_{i,B}(X_{p_i(B)}))_B.$$

For the duration of this proof we will denote $\overline{M}_{0,k}(\mathbb{C})$ by $\overline{M}_{0,k}$ and the family $\mathcal{G}(r,d)$ over $\overline{M}_{0,k}$ by $\mathcal{G}_k$. The above embeddings define an embedding

$$\prod_{A\subset_3[k]} \mathrm{Gr}(r,d)_A \times \prod_{i=1}^{k} \prod_{A\subset_3[|\lambda_i|+1]} \mathrm{Gr}(r,d)_A$$
$$\hookrightarrow \prod_{\substack{B\subset_3\mathbb{S}\\ \#p(B)=3}} \mathrm{Gr}(r,d)_B \times \prod_{i=1}^{k} \prod_{\substack{B\subset_3\mathbb{S}\\ b^\sharp=\{i,i,*\}}} \mathrm{Gr}(r,d)_B = \prod_{A\subset_3[\tilde{k}]} \mathrm{Gr}(r,d)_A.$$

This means we can realise $\mathcal{G}_k \times \prod_{i=1}^k \mathcal{G}_{|\lambda_i|+1}$ and thus $\mathcal{S}(\lambda_\bullet) \times \prod_{i=1}^k \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^{\mathsf{c}})$ as a subvariety of $\overline{M}_{0,\tilde{k}} \times \prod_{A\subset_3[\tilde{k}]} \mathrm{Gr}(r,d)_A$.

Now we show that $\mathcal{S}(\lambda_\bullet) \times \prod_{i=1}^k \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^{\mathsf{c}})$ coincides with $\mathcal{Y}$ when considered as a subvariety of $\overline{M}_{0,\tilde{k}} \times \prod_{A\subset_3[\tilde{k}]} \mathrm{Gr}(r,d)_A$. We will do this by checking fibres using Theorem 4.2.5. Fix a point $(C, C_i) \in M_{0,k} \times \prod_i M_{0,|\lambda_i|+1}$ and the corresponding point $\tilde{C} \in \overline{M}_{0,\tilde{k}}$. The irreducible components of $\tilde{C}$ are $C_1, C_2, \ldots, C_k, C$. Consider $B \subset_3 \mathbb{S}$ such that $\#p(B) = 3$ and as before set $A = p(B)$. Recall from Section 4.2.2 the definition of the maps $\phi_{1,A}$ and $\phi_{k+1,B}$. We have a commutative diagram of isomorphisms

$$\begin{tikzcd} & C_A \arrow[d, "\alpha_B"] \\ C \arrow[ur, "\phi_{1,A}"] \arrow[r, "\phi_{k+1,B}"'] & C_B. \end{tikzcd} \tag{4.6.2}$$

So $\phi_{k+1,B}$ factors through α_B. Recall that $\mathrm{Gr}(r,d)_C$ is defined as a closed subvariety of $\prod_{v(B)=C} \mathrm{Gr}(r,d)_B$. The condition $v(B) = C$ is equivalent to $\#p(B) = 3$, thus by (4.6.2) we have a string of inclusions

$$\mathrm{Gr}(r,d)_C \subset \prod_{A\subset_3[k]} \mathrm{Gr}(r,d)_A \subset \prod_{\substack{B\subset_3\mathbb{S}\\ \#p(B)=3}} \mathrm{Gr}(r,d)_B.$$

In the same way, $\phi_{i,B} = \alpha_{i,B} \circ \phi_{1,A}$ and the condition $B \subset_3 \mathbb{S}$ and $B^\sharp = \{i, i, *\}$ is equivalent to $v(B) = C_i$, so we have a string of inclusions

$$\mathrm{Gr}(r,d)_{C_i} \subset \prod_{A\subset_3[|\lambda_i|+1]} \mathrm{Gr}(r,d)_A \subset \prod_{\substack{B\subset_3\mathbb{S}\\ \#B^\sharp=\{i,i,*\}}} \mathrm{Gr}(r,d)_B.$$

In particular what we have shown is that the fibre of $\mathcal{G}_k \times \prod_i \mathcal{G}_{|\lambda_i|+1}$ over $\tilde{C}$ is $\mathrm{Gr}(r,d)_C \times \prod_i \mathrm{Gr}(r,d)_{C_i}$. Let $z = (z_1, z_2, \ldots, z_k)$ be the marked points on C and

$z^{(i)} = (z^{(i)}_1, z^{(i)}_2, \ldots, z^{(i)}_{|\lambda_i|+1})$ be the marked points on C_i. A direct application of Theorem 4.2.5 shows that both the fibre of $\mathcal{S}(\lambda_\bullet) \times \prod_{i=1}^k \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^{\mathsf{c}})$ and the fibre of $\mathcal{Y}$ over $\tilde{C}$ is

$$\Omega(\lambda_\bullet; z)_C \times \prod_{i=1}^{k} \Omega(\square^{|\lambda_i|}, \lambda_i^{\mathsf{c}}; z^{(i)}).$$

We could now repeat the analysis of fibres for any curve $\tilde{C} \in \overline{M}_{0,k} \times \prod_i \overline{M}_{0,|\lambda_i|+1}$ in exactly the same way, the only difference is it becomes a little more involved to keep track of the notation. Alternatively, since $\mathcal{S}(\lambda_\bullet) \times \prod_{i=1}^k \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^{\mathsf{c}})$ and $\mathcal{Y}$ are the closure of their restriction to $M_{0,k} \times \prod_i M_{0,|\lambda_i|+1}$ and since they coincide on this open set they must coincide everywhere. □

4.6.3 The CW-structure and wall crossing for general $\lambda_\bullet$

Theorem 4.6.5 ([Spe14, Theorem 8.2]). *The faces of maximal dimension in the CW-structure on $\mathcal{S}(\lambda_\bullet)$ are labelled by pairs (s, γ) of a circular ordering s and a decgd γ of shape $(\lambda_{s(1)}, \lambda_{s(2)} \ldots, \lambda_{s(k)})$.*

We leave it to the reader to consult [Spe14] for a rigorous proof of this fact however we will comment on how the results of Section 4.6.2 allow us to make this statement and produce the dual equivalence classes. We use the notation of Section 4.6.2. Let Θ be a $(k-3)$-associahedron in $\mathcal{S}(\lambda_\bullet)$. We consider the embedding

$$\Theta \times \textstyle\prod_{i=1}^k \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^C)(\mathbb{R}) \longhookrightarrow \mathcal{S}(\square^{\tilde{k}})(\mathbb{R}).$$

Since this is an embedding of CW-complexes $\Theta \times \prod_{i=1}^k \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^C)(\mathbb{R})$ must be contained in some $(\tilde{k}-3)$-associahedron $\tilde{\Theta}$ of $\mathcal{S}(\square^{\tilde{k}})(\mathbb{R})$. Let $(\tilde{s}, \tilde{\gamma})$ be the circular order (of $\tilde{k}$) and cylindrical growth diagram labelling $\tilde{\Theta}$ as per Proposition 4.6.2. Let τ be the unique order preserving bijection

$$\left\{ \tilde{s}(k_1^i) \,\middle|\, 1 \le i \le k \right\} \longrightarrow \{1, 2, \ldots, k\}, \quad \text{where } k_1^i = 1 + \sum_{j=1}^{i-1} |\lambda_j|,$$

then $s(i) = \tau \circ \tilde{s}(k_1^i)$. Let $m(i) = \left|\lambda_{s(i)}\right|$, then γ, the decgd labelling Θ is defined to be the reduction of $\tilde{\gamma}$ modulo m

Proposition 4.6.6 ([Spe14, Section 9]). *Suppose we have two neighbouring associahedra of $\mathcal{S}(\lambda_\bullet)$ labelled by the pairs (s, γ) and $(\hat{s}, \hat{\gamma})$ where $\hat{s}$ is obtained from s by reversing $s(p), s(p+1), \ldots, s(q-1)$. The dual equivalence cylindrical growth diagram $\hat{\gamma}$ is given by*

$$\hat{\gamma}_{ij} = \begin{cases} \gamma_{ij} & \text{if } [i,j] \cap [p,q] = \emptyset \text{ or } [p,q] \subseteq [i,j], \\ \gamma_{(p+q-j)(p+q-i)} & \text{if } [i,j] \subseteq [p,q], \end{cases} \tag{4.6.3}$$

with the dual equivalence classes α_{ij} and β_{ij} being similarly flipped inside the triangle south-west of the node (p, q). The same description of flipping triangles as in Figure 4.8 holds.

4.7 The MTV isomorphism

In this section we outline the work of Mukhin, Tarasov and Varchenko which describes the connection between the spectrum of Bethe algebras and Schubert calculus. Let $\lambda_\bullet = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a sequence of partitions and μ a partition such that $|\mu| = |\lambda_\bullet|$ and $\lambda_i, \mu \in \texttt{Part}(r)$. Choose an integer d such that $r(d-r) \geq |\lambda_\bullet|$.

Let $z = (z_1, z_2, \dots, z_n) \in X_n$. Recall from Section 3.1.3 that $A(\lambda_\bullet; z)_\mu$ is the Bethe algebra associated to $L(\lambda_\bullet; z)_\mu^{\mathsf{sing}}$ and $\mathcal{A}(\lambda_\bullet, z)_\mu$ is the spectrum of $A(\lambda_\bullet; z)_\mu$. Let $\chi \in \mathcal{A}(\lambda_\bullet, z)_\mu$ be a closed point which we identify with a function $\chi : A(\lambda_\bullet; z)_\mu \longrightarrow \mathbb{C}$. Define the differential operator on $\mathbb{C}[u]$,

$$\mathcal{D}^\chi = \partial^r + \sum_{i=1}^{r} B_i^\chi(u) \partial^{r-i},$$

where $B_i^\chi(u) = \sum_{s=0}^{\infty} \chi(B_{is}) u^{-s-1}$. By [MV04, Lemma 5.6] the kernel of this differential operator, $X_\chi = \ker \mathcal{D}^\chi$ is an r-dimensional subspace of $\mathbb{C}[u]$. Let $\Omega^\circ(\lambda_\bullet, \mu^{\mathsf{c}}; z, \infty)$ be the scheme theoretic intersection

$$\Omega^\circ(\lambda_\bullet, \mu^{\mathsf{c}}; z, \infty) = \Omega^\circ(\lambda_1; z_1) \cap \Omega^\circ(\lambda_2; z_2) \cap \ldots \cap \Omega^\circ(\lambda_n; z_n) \cap \Omega^\circ(\mu^{\mathsf{c}}; \infty).$$

Theorem 4.7.1 ([MTV09a, Theorem 5.13]). *There is an isomorphism of schemes $\kappa_z : \mathcal{A}(\lambda_\bullet, z)_\mu \longrightarrow \Omega^\circ(\lambda_\bullet, \mu^{\mathsf{c}}; z, \infty)$ such that $\kappa_z(\chi) = X_\chi$.*

Corollary 4.7.2. *The schemes $\Omega^\circ(\lambda_\bullet, \mu^{\mathsf{c}}; z)$ and $\Omega(\lambda_\bullet, \mu^{\mathsf{c}}; z)$ are equal as subschemes of $\mathrm{Gr}(r, d)$ for any $z \in M_{0,n+1}(\mathbb{C})$.*

Proof. We clearly have an inclusion of schemes $\Omega^\circ(\lambda_\bullet, \mu^{\mathsf{c}}; z) \subset \Omega(\lambda_\bullet, \mu^{\mathsf{c}}; z)$. By the remarks in section 4.1.3, the number of scheme theoretic points in $\Omega(\lambda_\bullet, \mu^{\mathsf{c}})$ is $c^\mu_{\lambda_\bullet}$. Theorem 4.7.1 combined with Theorem 3.1.10 (i) says that there are already $c^\mu_{\lambda_\bullet}$ scheme theoretic points in $\Omega^\circ(\lambda_\bullet, \mu^{\mathsf{c}}; z)$. □

4.7.1 An auxillary Bethe algebra

Later, we will want a global version of Theorem 4.7.1, i.e. we would like to show $\mathcal{A}(\lambda_\bullet)_\mu$ is isomorphic to $\Omega(\lambda_\bullet, \mu^{\mathsf{c}})$ as families over $M_{0,n+1}(\mathbb{C})$. In order to do this we follow Mukhin, Tarasov and Varchenko [MTV09a], and define a global version of the Bethe algebra. Consider the tensor product

$$\mathcal{V}_n = V^{\otimes n} \otimes \mathbb{C}[x_1, x_2, \dots, x_n],$$

of the n-fold tensor product of the vector representation with a polynomial ring. This has the structure of a $\mathfrak{gl}_r[t]$-representation given by

$$xt^s \cdot f(x_1, x_2, \dots, x_n) v_1 \otimes v_2 \otimes \dots \otimes v_n$$
$$= f(x_1, x_2, \dots, x_n) \sum_{a=1}^{n} z_a^s v_1 \otimes \dots \otimes x \cdot v_a \otimes \dots \otimes v_n.$$

it also has the structure of an S_n-representation given by

$$\sigma \cdot f(x_1, x_2, \dots, x_n) v_1 \otimes v_2 \otimes \dots \otimes v_n$$
$$= f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \dots \otimes v_{\sigma^{-1}(n)}.$$

In particular, these two actions commute so $\mathfrak{gl}_r[t]$ acts on the subspace of invariant vectors for the action of the symmetric group $\mathcal{V}_n^{S_n} \subset \mathcal{V}_n$. For a partition μ of n, denote the Bethe algebra associated to $[\mathcal{V}_n^{S_n}]_\mu^{\mathsf{sing}}$ by A_μ. Denote the spectrum of the algebra A_μ by $\mathcal{A}_\mu$.

Lemma 4.7.3. *The $\mathfrak{gl}_r[t]$-module $L(\lambda_\bullet; z)_\mu^{\mathsf{sing}}$ is a subquotient of the module $[\mathcal{V}_{\tilde{n}}^{S_n}]_\mu^{\mathsf{sing}}$. In particular, $A(\lambda_\bullet; z)_\mu$ is a quotient of A_μ.*

Proof. In [MTV09a, Corollary 2.4 and Lemma 2.13] the first statement is proven and the second is a consequence of a general fact remarked at the beginning of [MTV09a, Section 5.4] which we explain here. If B is an algebra and M and N are B-modules so that M is a subquotient of N, let B_M and B_N be the image of B in $\mathrm{End}(M)$ and $\mathrm{End}(N)$ respectively. We claim that there is a surjection $B_N \to B_M$.

The module M is a subquotient of N so there is a submodule $L \subset N$ such that $M \subset N/L$. Define a map $\alpha \colon B_N \longrightarrow \mathrm{End}(M)$ by $\alpha(\varphi)(m) = \varphi(\tilde{m}) + L$, for any lift $\tilde{m} \in N$ of $m \in M$. To see this is well defined we need to check two things, first that $\alpha(\varphi)(m) \in M$ and second that it does not depend on the choice of lift. To see that $\alpha(\varphi)(m) \in M$, note that $\tilde{M} \subset N$, the preimage of the canonical projection to N/L is a submodule, so $\varphi(\tilde{m}) \in \tilde{M}$, thus $\varphi(\tilde{m}) + L \in M$. Now suppose $\hat{m}$ is another lift of m, so $\tilde{m} - \hat{m} \in L$. Since L is a B-submodule and $\varphi \in B_N$, $\varphi(\tilde{m} - \hat{m}) = \varphi(\tilde{m}) - \varphi(\hat{m}) \in L$. Let ρ_M and ρ_N be the representations afforded by M and N, we check that $\alpha \circ \rho_N = \rho_M$ which shows α surjects onto B_M. Let $m \in M$, then

$$(\alpha \circ \rho_N(b))(m) = \rho_N(b)(\tilde{m}) + L = \rho_M(b)(\tilde{m} + L) = \rho_M(b)(m),$$

by the definition of the action of B on quotient modules. $\square$

This lemma, in particular means we can think of $\mathcal{A}(\lambda_\bullet; z)_\mu$ as a subscheme of $\mathcal{A}_\mu$. Mukhin, Tarasov and Varchenko actually proved Theorem 4.7.1 by first relating $\mathcal{A}_\mu$ to the Schubert cell $\Omega^\circ(\mu^{\mathsf{c}}; \infty)$. Note that $\Omega^\circ(\lambda_\bullet, \mu^{\mathsf{c}}; z, \infty)$ is a subscheme of $\Omega^\circ(\mu^{\mathsf{c}}; \infty)$. Let ι be the corresponding inclusion.

Theorem 4.7.4 ([MTV09a, Theorem 5.3]). *There exists an isomorphism of schemes* $\tilde{\kappa} : \mathcal{A}_\mu \longrightarrow \Omega^\circ(\mu^c; \infty)$ *such that* $\tilde{\kappa} \circ \iota = \kappa_z$.

Said more explicitly, Theorem 4.7.4 says the isomorphism $\tilde{\kappa}$ restricts to the isomorphism κ_z of the subschemes $\mathcal{A}(\lambda_\bullet, z)_\mu$ and $\Omega^\circ(\lambda_\bullet, \mu^c; z, \infty)$. This allows us to prove the family version of this result.

Corollary 4.7.5. *The Bethe spectrum* $\mathcal{A}(\lambda_\bullet)_\mu$ *and the intersection* $\Omega(\lambda_\bullet, \mu^c)$ *are isomorphic as families over* $M_{0,n+1}(\mathbb{C})$.

Proof. First note that $\Omega(\lambda_\bullet, \mu^c) = \Omega^\circ(\lambda_\bullet, \mu^c) \subset \Omega^\circ(\mu^c; \infty)$ by Corollary 4.7.2. By Theorem 4.7.4 we have an isomorphism of trivial families

$$\tilde{\kappa} \times \mathrm{id} : \mathcal{A}_\mu \times M_{0,n+1}(\mathbb{C}) \longrightarrow \Omega^\circ(\mu^c; \infty) \times M_{0,n+1}(\mathbb{C}).$$

Identifying $\mathcal{A}(\lambda_\bullet)_\mu$ and $\Omega(\lambda_\bullet, \mu^c)$ as subfamilies, Theorem 4.7.4 also says this isomorphism restricts to isomorphisms of the fibres of $\pi_{\lambda_\bullet, \mu}$ and $\vartheta^\circ_{\lambda_\bullet, \mu^c}$ over closed points. Since both of these families are finite and flat, Proposition C.2.4 allows us to conclude that we have a global isomorphism. □

4.8 The algebraic Bethe Ansatz

Along with Bethe algebras and Schubert intersections, there is a third important player in the story, the critical points of the *master function*. The relationship between these three objects has been studied extensively by Mukhin, Tarasov and Varchenko, see for example [MTV12]. Critical points have a labelling by standard tableaux in a similar way to points in the spectrum of Bethe algebras (see Section 3.4.2), this is described by Marcus [Mar10] and for the sake of convenience we recall the proof of this result. This will be vital in proving Theorem B.

4.8.1 Notation

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{gl}_r$ (so $\mathfrak{h}$ is the algebra of diagonal matrices). Let $(\cdot, \cdot)$ denote the trace form on $\mathfrak{gl}_r$ (i.e. the normalised Killing form). Let $h_i = e_{ii} - e_{i+1,i+1}$ for $i = 1, \ldots, r-1$. Let $\varepsilon_i \in \mathfrak{h}^*$ be the dual vector to e_{ii} and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ the dual vector to h_i. With this notation the trace form, transported to $\mathfrak{h}^*$ has the following values,

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \tag{4.8.1}$$

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1. \end{cases} \tag{4.8.2}$$

We identify a partition λ with at most r parts, with the $\mathfrak{gl}_r$-weight $\sum \lambda^{(i)} \varepsilon_i$.

4.8.2 The master function and critical points

We take our usual setup, let $z = (z_1, z_2, \ldots, z_n)$ be an n-tuple of complex variables, let $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a sequence of partitions and let μ be a partition such that $|\mu| = |\lambda_\bullet|$. We also require that there exist non-negative integers l_i such that $\mu = \sum \lambda^{(s)} - \sum_{i=1}^{r-1} l_i\alpha_i$. This last requirement ensures μ appears as a weight in $L(\lambda_\bullet)$. We let $t_i^{(j)}$ be a set of complex variables for $i = 1, 2, \ldots, r-1$ and $j = 1, 2, \ldots, l_i$.

Definition 4.8.1. The *master function* is the rational function

$$\Phi(\lambda_\bullet, \mu; z, t) = \Phi(z, t) =$$

$$\prod_{1\le a<b\le n} (z_a - z_b)^{(\lambda_a,\lambda_b)} \prod_{a=1}^{n}\prod_{i=1}^{r-1}\prod_{j=1}^{l_i}(z_a - t_i^{(j)})^{-(\lambda_a,\alpha_i)} \prod_{(i,a)<(j,b)} (t_i^{(a)} - t_j^{(b)})^{(\alpha_i,\alpha_j)}.$$

The ordering $(i, a) < (j, b)$ is taken lexicographically. Let $S = \log \Phi$. The *Bethe ansatz equations* are given by the system of rational functions,

$$\frac{\partial S}{\partial t_i^{(j)}} = \frac{\partial}{\partial t_i^{(j)}} \log \Phi(z, t) = 0 \quad \text{for } i = 1, 2, \ldots, r-1 \text{ and } j = 1, 2, \ldots, l_i. \tag{4.8.3}$$

A solution to the Bethe ansatz equations is called a *critical point*. We say a critical point $t = (t_i^{(j)})$ is *nondegenerate* if the Hessian of S,

$$\mathrm{Hess}(S) = \det\left(\frac{\partial S}{\partial t_i^{(a)}\partial t^{(b)}}\right)_{(i,a),(j,b)},$$

evaluated at t is invertible.

Let $m = l_1 + l_2 + \ldots + l_{r-1}$. The Bethe ansatz equations are rational functions on $X_n \times \mathbb{C}^m$, regular away from the finite collection of hyperplanes given by $t_i^{(a)} - t_j^{(b)} = 0$. Let $\tilde{\mathcal{C}}(\lambda_\bullet)_\mu$ denote the vanishing set of the Bethe ansatz equations, considered as a family over X_n. Let $S_\mathfrak{l}$ be the product of symmetric groups $S_{l_1} \times S_{l_2} \times \cdots S_{l_{r-1}} \subset S_m$, which acts on $\mathbb{C}^m$ by permuting the coordinates $t_i^{(j)}$ with the same lower index. Using (4.8.1),

$$\prod_{(i,a)<(j,b)} (t_i^{(a)} - t_j^{(b)})^{(\alpha_i,\alpha_j)} = \prod_{i=1}^{r-2}\prod_{a=1}^{l_i}\prod_{b=1}^{l_{i+1}}(t_i^{(a)} - t_{i+1}^{(b)})^{-1} \prod_{i=1}^{r-1} \prod_{1\le a<b\le l_i} (t_i^{(a)} - t_i^{(b)})^2.$$

Thus $\Phi(z, t)$ is invariant under the action of $S_\mathfrak{l}$. The quotient $\tilde{\mathcal{C}}(\lambda_\bullet)_\mu/S_\mathfrak{l}$ will be denoted $\mathcal{C}(\lambda_\bullet)_\mu$ and the open subset of nondegenerate critical points $\mathcal{C}(\lambda_\bullet)_\mu^{\mathsf{nondeg}}$. Let $\mathbb{P}^\infty = \mathbb{P}\mathbb{C}[u]$ be the infinite dimensional projective space associated to the polynomial ring. We think of $\mathbb{P}^\infty$ as the space of monic polynomials. For any $a \in \mathbb{Z}_{\ge 0}$, there is an embedding $\mathbb{C}^a/S_a \hookrightarrow \mathbb{P}^\infty$ given by sending the orbit of a point $(t_1, \ldots, t_a) \in \mathbb{C}^a$ to the unique monic polynomial of degree a, with roots $t_1, \ldots, t_a$. We will identify $\mathcal{C}(\lambda_\bullet)_\mu$ with its image in $X_n \times (\mathbb{P}^\infty)^{r-1}$ and denote the tuple of monic polynomials associated to a critical point $t = (t_i^{(j)})$ by $y^t = (y_1^t, \ldots, y_{r-1}^t)$. To be clear, this means if $t = (t_i^{(j)})$ is a solution of the Bethe ansatz equations (4.8.3), then y_i^t is a monic polynomial in u, with roots $t_i^{(1)}, t_i^{(2)}, \ldots, t_i^{(l_i)}$. Let $p_{\lambda_\bullet,\mu}$ denote the projection $\mathcal{C}(\lambda_\bullet)_\mu \to X_n$. Denote the fibre of $\mathcal{C}(\lambda_\bullet)_\mu$ over $z \in X_n$ by $\mathcal{C}(\lambda_\bullet; z)_\mu$.

Theorem 4.8.2 ([MTV12, Theorem 6.1])**.** *There exists a function, called the* universal weight function, $\omega : X_n \times \mathbb{C}^m/S_{\mathrm{l}} \longrightarrow L(\lambda_\bullet)_\mu$ *such that, for a critical point* $t = (t_i^{(j)})$ *in* $\mathcal{C}(\lambda_\bullet; z)_\mu$*, then*

(i) $\omega(z, t) \in L(\lambda_\bullet)_\mu^{\mathsf{sing}}$,

(ii) the critical point t *is nondegenerate if and only if* $\omega(z, t)$ *is nonzero,*

(iii) if $t' \in \mathcal{C}(\lambda_\bullet; z)_\mu$ *is a critical point distinct from* t*, and both are nondegenerate then* $\omega(z, t)$ *and* $\omega(z, t')$ *are linearly independent,*

(iv) $\omega(z, t)$ *is a simultaneous eigenvector for* $A(\lambda_\bullet; z)_\mu$*, and*

(v) the eigenvalue of $H_a(z)$ *acting on* $\omega(z, t)$ *is*

$$\frac{\partial S}{\partial z_a}(z, t).$$

4.8.3 Examples of the Bethe ansatz equations

Below are some examples of the Bethe ansatz equations in simple cases. Explicitly, in full generality, the Bethe ansatz equations are

$$\frac{\partial S}{\partial t_i^{(j)}} = -\sum_{a=1}^{n} (\alpha_i, \lambda_a) \frac{1}{t_i^{(j)} - z_a} + \sum_{(k,a)\neq(i,j)} (\alpha_i, \alpha_k) \frac{1}{t_i^{(j)} - t_k^{(a)}} = 0. \tag{4.8.4}$$

Example 4.8.3. In the case $n = 1$, with $\lambda_\bullet = (\lambda)$, the only choice for μ is $\mu = \lambda$. Thus $l_i = 0$ for all i, the variable t is simply an empty variable. The master function becomes $\Phi(z, t) = 1$. The Bethe ansatz equations in this case are vacuously satisfied and there is a single unique critical point $t_\emptyset$ (the empty critical point). The polynomial $y_i^{t_\emptyset}$ is the unique monic polynomial with no roots, i.e. the constant polynomial 1. Thus $\mathcal{C}(\lambda; z)_\mu \subset (\mathbb{P}^\infty)^{r-1}$ is a single point.

Example 4.8.4. In this thesis we will be primarily interested in the case $\lambda_i = \square = \varepsilon_1$ for all i. In this case $|\mu| = n$. Since the highest possible weight in $V^{\otimes n}$ is $(n) = n\varepsilon_1$, the integer l_i is the number of boxes in μ sitting strictly below the i^{th} row.

In this case $(\varepsilon_1, \varepsilon_1) = 1$, and $(\alpha_i, \varepsilon_1) = \delta_{1,i}$. Thus the master function becomes

$$\Phi(z,t) = \prod_{1\le a<b\le n} (z_a - z_b) \prod_{a=1}^{n}\prod_{j=1}^{l_1}(t_j^{(1)} - z_a)^{-1} \prod_{i=1}^{r-2}\prod_{a=1}^{l_i}\prod_{b=1}^{l_{i+1}}(t_i^{(a)} - t_{i+1}^{(b)})^{-1}$$
$$\prod_{i=1}^{r-1}\prod_{1\le a<b\le l_i} (t_i^{(a)} - t_i^{(b)})^2.$$

Example 4.8.5. Next consider the special case when $n = 2$, so $\lambda_\bullet = (\lambda_1, \lambda_2)$ for some partitions λ_1 and λ_2. Let $z = (z_1, z_2)$. Make a change of variables

$$s_i^{(j)} = \frac{t_i^{(j)} - z_1}{z_2 - z_1}.$$

In these new variables, the Bethe ansatz equations become

$$0 = \frac{\partial S}{\partial s_i^{(j)}}\frac{\partial s_i^{(j)}}{\partial t_i^{(j)}} = \left(-\frac{(\lambda_1, \alpha_i)}{s_i^{(j)}} - \frac{(\lambda_2, \alpha_i)}{s_i^{(j)} - 1} + \sum_{(k,a)\ne(i,j)} \frac{(\alpha_i, \alpha_k)}{s_i^{(j)} - s_k^{(a)}} \right) \frac{1}{z_2 - z_1},$$

which can be rearranged to

$$\frac{(\lambda_1, \alpha_i)}{s_i^{(j)}} + \frac{(\lambda_2, \alpha_i)}{s_i^{(j)} - 1} = \sum_{(k,a)\ne(i,j)} \frac{(\alpha_i, \alpha_k)}{s_i^{(j)} - s_k^{(a)}}, \tag{4.8.5}$$

and thus do not depend on z_1 and z_2. These are the *transformed bethe ansatz equations.* The set of (orbits of) solutions of (4.8.5) is denoted $\mathcal{S}(\lambda_1, \lambda_2)_\mu$.

Consider the special case, when $\lambda_1 = \lambda$ and $\lambda_2 = \Box$. By the Pieri rule, for the μ-weight space to be nonzero, μ must be obtained from λ by adding a single box. Suppose the box is added in row e. Then $l_i = 1$ for $i = 1, 2, \ldots, e-1$ and $l_i = 0$ otherwise. Setting $s_i = s_i^{(1)}$ for $i = 1, 2, \ldots, e-1$, equation (4.8.5) can be rewritten,

$$\frac{(\lambda_1, \alpha_i)}{s_i} + \frac{\delta_{1i}}{s_i - 1} = \frac{\delta_{i1} - 1}{s_i - s_{i-1}} + \frac{\delta_{(i+1)e} - 1}{s_i - s_{i+1}}. \tag{4.8.6}$$

Proposition 4.8.6 ([Mar10, Lemma 7.2])**.** *There is a unique solution to the transformed Bethe ansatz equations (4.8.6), that is $\mathcal{S}(\lambda, \Box)_\mu$ is a single point. In particular*

$$s_1 = 1 - \left(\lambda^{(1)} - c\right)^{-1}, \tag{4.8.7}$$

where c is the content (see Section 3.2.7) of the box $\mu\backslash\lambda$.

4.8.4 Analytic continuation of critical points

In order to discuss the asymptotic properties of critical points, it is important to be able to choose a critical point $t \in \mathcal{C}(\lambda_\bullet; z)_\mu$ and perturb the parameters z to obtain new critical points depending locally on the variables z. Essentially, this means we would like to apply the inverse function theorem to the morphism

$$p_{\lambda_\bullet,\mu} : \mathcal{C}(\lambda_\bullet)_\mu \longrightarrow X_n,$$

which we can do if the morphism is étale. We use the definition of étale given in [GW10, Definition 6.14] as a smooth morphism of relative dimension zero.

Proposition 4.8.7. *The restriction of the morphism $p_{\lambda_\bullet,\mu}$ to $\mathcal{C}(\lambda_\bullet)_\mu^{\mathsf{nondeg}}$ is étale.*

Proof. By [MV04, Theorem 5.13] there can be at most $c^\mu_{\lambda_\bullet} = \dim L(\lambda_\bullet)_\mu^{\mathsf{sing}}$ closed points in $\mathcal{C}(\lambda_\bullet; z)_\mu$. This means $p_{\lambda_\bullet,\mu}$ is a quasi-finite morphism.

Let $S_{(i,j)}$ be the partial derivative of S at $t_i^{(j)}$ and the $R = \mathbb{C}[X_n]$. Note that $\mathbb{C}[\mathcal{C}(\lambda_\bullet)_\mu] = R[t_j^{(i)}]/(S_{i,j})$ where the generators and relations range over all possible i, j. By definition, $p_{\lambda_\bullet,\mu}$ is smooth of relative dimension zero at $t \in \mathcal{C}(\lambda_\bullet)_\mu$ if the matrix

$$\left(\frac{\partial S_{(i,j)}}{\partial t_k^{(a)}}\right)_{(i,j),(a,k)} = \mathrm{Hess}(S),$$

is square and nonsingular which holds by definition if and only if $t \in \mathcal{C}(\lambda_\bullet)_\mu^{\mathsf{nondeg}}$. $\square$

Let X_n^{nondeg} be the image of $\mathcal{C}(\lambda_\bullet)_\mu^{\mathsf{nondeg}}$ under $p_{\lambda_\bullet,\mu}$. Since $p_{\lambda_\bullet,\mu}$ is étale, for any nondegenerate critical point $t \in \mathcal{C}(\lambda_\bullet; x)_\mu^{\mathsf{nondeg}}$ there exits a local (holomorphic) section of $p_{\lambda_\bullet,\mu}$, which we denote $t(z)$, such that $t(x) = t$. We can then analytically continue along any path in X_n^{nondeg}.

4.8.5 Asymptotics of critical points and Marcus' labelling

Later, we will need a result about the asymptotics of critical points as we send the parameters to infinity. Reshetikhin and Varchenko [RV95] explain how to glue two nondegenerate critical points to obtain a critical point for a larger master function with parameters $z = (z_1, z_2, \ldots, z_{n+k})$. Their theorem allows one to track the analytic continuation of this new critical point as we send the parameters $z_{n+1}, \ldots, z_{n+k}$ to infinity and shows that asymptotically we recover the two critical points we started with. The set up for the theorem is the following data, two sequences of partitions,

- $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, and
- $\lambda'_\bullet = (\lambda'_1, \lambda'_2, \ldots, \lambda'_k)$.

Three additional partitions,

- $\nu = \sum_i \lambda_i - \sum_j a_j\alpha_j$,
- $\nu' = \sum_i \lambda'_i - \sum_j b_j\alpha_j$, and

- $\mu = \nu + \nu' - \sum_j c_j\alpha_j = \sum_i \lambda_i + \sum_i \lambda'_i - \sum_j (a_j + b_j + c_j)\alpha_j$,

for nonnegative integers a_j, b_j and c_j. Two nondegenerate critical points

- $u = (u_i^{(j)}) \in \mathcal{C}(\lambda_\bullet; z)_\nu^{\mathsf{nondeg}}$, and
- $v = (v_i^{(j)}) \in \mathcal{C}(\lambda'_\bullet; x)_{\nu'}^{\mathsf{nondeg}}$,

for complex points, $z = (z_1, z_2, \ldots, z_n)$ and $x = (x_1, x_2, \ldots, x_k)$; and finally a solution, $s = (s_i^{(j)}) \in \mathcal{S}(\nu, \nu')_\mu$, to the transformed Bethe ansatz equations (4.8.5).

Theorem 4.8.8 ([RV95, Theorem 6.1])**.** *In the limit when $z_{n+1}, z_{n+2}, \ldots, z_{n+k}$ are sent to ∞ in such a way that $z_{n+i} - z_{n+1}$ remain finite for $i = 1, 2, \ldots, k$, there exists a unique nondegenerate critical point $t = (t_i^{(j)}) \in \mathcal{C}(\lambda_\bullet, \lambda'_\bullet; z)_\mu^{\mathsf{nondeg}}$ such that asymptotically, the critical point has the form*

$$t_i^{(j)}(z) = \begin{cases} u_i^{(j)}(z_1, \ldots, z_n) + O(z_{n+1}^{-1}) & \text{if } 1 \le j \le a_i, \\ s_i^{(j)} z_{n+1} + O(1) & \text{if } a_i < j \le a_i + c_i, \\ v_i^{(j)}(x_1, \ldots, x_k) + z_{n+1} + O(z_{n+1}^{-1}) & \text{if } a_i + c_i < j \le a_i + b_i + c_i, \end{cases}$$

where $x_i = z_{n+i} - z_{n+1}$ for $i = 1, 2, \ldots, k$.

Corollary 4.8.9. *Let t, u, v and s be as in Theorem 4.8.8. Taking a limit $z_{n+i} \to \infty$ such that $z_{n+i} - z_{n+1}$ is bounded, (which we denote $\lim_{z\to\infty}$) we have*

$$\lim_{z\to\infty} y^t = y^u.$$

Proof. This is a direct application of Theorem 4.8.8 to the definition of y^t. □

We restrict our attention to critical points for $z = (z_1, z_2, \ldots, z_n)$ and n-tuple of distinct real numbers such that $z_1 < z_2 < \ldots < z_n$. In the limit when $z_1, z_2, \ldots, z_n \to \infty$ such that $z_i = o(z_{i+1})$, Marcus, [Mar10], describes a method to label critical points in $\mathcal{C}(\square^n; z)_\mu$ by standard μ-tableaux. Marcus' theorem is recalled below, along with the proof. Recall, if $T \in \mathtt{SYT}(\mu)$ then $T|_{n-1}$ is the tableaux obtained by removing the box containing n from T.

Theorem 4.8.10 ([Mar10, Theorem 6.1])**.** *Given a standard tableaux, T, of shape μ, there is a unique critical point $t_T \in \mathcal{C}(\square^n; z)_\mu$ such that, if $y^T = y^{t_T}$,*

$$\lim_{z_n\to\infty} y^T = y^{T|_{n-1}}, \tag{4.8.8}$$

and taking the limit $z_1, z_2, \ldots, z_n$ such that $|z_i| << |z_{i+1}|$, asymptotically

$$z_a \frac{\partial S}{\partial z_a} \sim c_T(i) + O(z_i^{-1}). \tag{4.8.9}$$

Proof. We will prove this by induction on n. For $n = 1$, the only partition is $\square$ and the only tableaux is $T = \boxed{1}$. From Example 4.8.3 we know $\mathcal{C}(\square; z)_\square$ contains a unique critical point, the empty critical point $t_\emptyset$ and we simply set $t_T = t_\emptyset$. Thus $y^T = 1$. The equations (4.8.8) and (4.8.9) are vacuously satisfied.

For general n, we will use Theorem 4.8.8 to inductively build a critical point corresponding to $T \in \mathtt{SYT}(\mu)$. Let $\lambda = \mathrm{sh}(T|_{n-1})$, the partition obtained by removing the box labelled n in T, from μ. By induction, there is a unique critical point $t_{T|_{n-1}} \in \mathcal{C}(\square^{n-1}; z_1, \ldots, z_{n-1})_\lambda$. To build a critical point in $\mathcal{C}(\square^n; z)_\mu$, we need to fix a critical point in $\mathcal{C}(\square; 0)_\square$, and a transformed critical point in $\mathcal{S}(\lambda, \square)_\mu$. The former contains only the empty critical point and the latter contains a unique point $s = (s_1, \ldots, s_{e-1})$ (where e is the row containing n in T) by Proposition 4.8.6, where

$$s_1 = 1 - \Big(\lambda^{(1)} - c_T(n)\Big)^{-1}. \tag{4.8.10}$$

Thus, given $t_{T|_{n-1}}$, and the data of where to add an n^{th} box to $T|_{n-1}$, we obtain by Theorem 4.8.8 a unique critical point $t_T \in \mathcal{C}(\square^n; z)_\mu$. By Corollary 4.8.9 we obtain (4.8.8).

All that is left to do is to prove (4.8.9). This will also be done by induction. We need to investigate the eigenvalues

$$z_a \frac{\partial S}{\partial z_a} = \sum_{b \neq a} \frac{z_a}{z_a - z_b} - \sum_{j=1}^{|\mu| - \mu^{(1)}} \frac{z_a}{z_a - t_1^{(j)}},$$

of the operators $z_a H_a(z)$. Suppose first that $a < n$, then

$$z_a \frac{\partial S}{\partial z_a} = \sum_{\substack{b \neq a \\ b < n}} \frac{z_a}{z_a - z_b} - \sum_{j=1}^{|\mu| - \mu^{(1)} - \delta_{e>1}} \frac{z_a}{z_a - t_1^{(j)}} + \frac{z_a}{z_a - z_n} - \frac{\delta_{e>1} z_a}{z_a - t_1^{(|\mu| - \mu^{(1)})}}.$$

But in the limit $z_a/(z_a - z_n) \sim 0$ and by Theorem 4.8.8 $t_1^{(|\mu| - \mu^{(1)})} \sim s_1 z_n + O(1)$ so

$$\begin{aligned} z_a \frac{\partial S}{\partial z_a} &\sim \sum_{\substack{b \neq a \\ b < n}} \frac{z_a}{z_a - z_b} - \sum_{j=1}^{|\mu| - \mu^{(1)} - \delta_{e>1}} \frac{z_a}{z_a - t_1^{(j)}} \\ &= \sum_{\substack{b \neq a \\ b < n}} \frac{z_a}{z_a - z_b} - \sum_{j=1}^{|\lambda| - \lambda^{(1)}} \frac{z_a}{z_a - t_1^{(j)}} \\ &= z_a \frac{\partial S'}{\partial z_a}, \end{aligned}$$

where $S' = S(\square^{n-1}, \lambda; z_1, \ldots, z_{n-1})$ is the logarithm of the master function for the weight λ. Thus by induction (4.8.9) holds for $a < n$.

Now we need to check (4.8.9) for $a = n$. This turns out to be a simple calculation. By Theorem 4.8.8

$$\begin{aligned} z_n \frac{\partial S}{\partial z_n} &= \sum_{b<n} \frac{z_n}{z_n - z_b} - \sum_{j=1}^{|\mu|-\mu^{(1)}} \frac{z_n}{z_n - t_1^{(j)}} \\ &\sim (n-1) - (|\mu| - \mu^{(1)} - \delta_{e>1}) - \frac{\delta_{e>1}}{1-s_1} + O(z_n^{-1}) \\ &= \mu^{(1)} - \delta_{e1} - \frac{\delta_{e>1}}{1-s_1} + O(z_n^{-1}). \end{aligned}$$

Using (4.8.10),

$$z_n \frac{\partial S}{\partial z_n} \sim \mu^{(1)} - \delta_{e1} - \delta_{e>1}\left(\lambda^{(1)} - c_T(n)\right) + O(z_n^{-1}).$$

If $e > 1$ then $\mu^{(1)} = \lambda^{(1)}$, and if $e = 1$ then $c_T(n) = \mu^{(1)} - 1$ so the Theorem is proved. □

4.8.6 The coordinate map

This section describes the relationship between critical points for the master function and Schubert intersections. Let $g_1(u), g_2(u), \ldots, g_k(u)$ be a collection of polynomials in the variable u. Recall the *Wronskian* is the determinant

$$\mathrm{Wr} = \det \begin{pmatrix} g_1(u) & g_2(u) & \cdots & g_k(u) \\ g_1'(u) & g_2'(u) & \cdots & g_k'(u) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(n-1)}(u) & g_2^{(n-1)}(u) & \cdots & g_k^{(n-1)}(u) \end{pmatrix}.$$

Lemma 4.8.11. *Up to a scalar factor, the Wronskian* $\mathrm{Wr}(g_1(u), \ldots, g_k(u))$*, depends only on the subspace of* $\mathbb{C}[u]$ *spanned by the polynomials* $g_i(u)$*, and is zero if and only if the polynomials are linearly dependant.*

Proof. The result follows from the fact that the derivative is a linear operation, and that the determinant can at most be multiplied by a scalar after applying column operations. □

Let $X \in \mathrm{Gr}(r, d)$. Since $\mathrm{Gr}(r, d)$ is paved by Schubert cells $X \in \Omega^\circ(\mu^c; \infty)$ for some unique partition μ. By definition X is an r-dimensional vector space of polynomials in the variable u, of degree less than d. Let l_i be the number of boxes below the i^{th} row in μ (c.f Example 4.8.4). Set $d_i = \mu_i + r - 1$. We can choose a ordered basis $f_1(u), f_2(u), \ldots, f_r(u)$ of monic polynomials with descending degrees d_i. Consider the polynomials

$$y_a(u) = \mathrm{Wr}(f_{a+1}(u), \ldots, f_r(u)), \quad a = 0, 1, \ldots, r-1.$$

The polynomial y_a has degree l_a. Denote its roots by $t_a^{(1)}, t_a^{(2)}, \ldots, t_a^{(l_a)}$. The polynomial $y_a(u)$ determines a point in $\mathbb{P}^\infty$. The following lemma demonstrates the polynomials $y_a(u) \in \mathbb{P}^\infty$ depend only on X and not the basis chosen.

Lemma 4.8.12. *Suppose $\{f_i(u)\}$ and $\{f_i'(u)\}$ are two bases of X of monic polynomials with descending degrees. Then*

$$\mathrm{Wr}(f_{a+1}(u), \ldots, f_r(u)) = \alpha\, \mathrm{Wr}(f_{a+1}'(u), \ldots, f_r'(u))$$

for some scalar $\alpha \in \mathbb{C}$.

Proof. We use the fact that the descending sequence $d_1 > d_2 > \ldots > d_r$ of degrees for any such basis is determined entirely by the partition μ. That is, $\deg f_i(u) = \deg f_i'(u) = d_i$. By Lemma 4.8.11 the Wronskian $\mathrm{Wr}(g_1, \ldots, g_k)$ is determined by the space spanned by the polynomials $g_1, \ldots, g_k$, we must prove

$$\mathbb{C}\left\{f_{a+1}(u), \ldots, f_r(u)\right\} = \mathbb{C}\left\{f_{a+1}'(u), \ldots, f_r'(u)\right\}. \tag{4.8.11}$$

Since both bases span X, $f_a'(u) = \alpha_1 f_1(u) + \alpha_2 f_2(u) + \ldots + \alpha_r f_r(u)$ for some complex numbers α_i. But the degrees of the f_i are strictly descending so $\alpha_i = 0$ for $i > a$. Hence $f_a'(u) \in \mathbb{C}\{f_a(u), \ldots, f_r(u)\}$. By induction (4.8.11) must be true. $\square$

Definition 4.8.13. The map $\theta\colon \mathrm{Gr}(r,d) \longrightarrow (\mathbb{P}^\infty)^r$ defined by

$$\theta(X) = (y_a)_{a=0}^{r-1} = (\mathrm{Wr}(f_{a+1}(u), \ldots, f_r(u)))_{a=0}^{r-1}$$

for some choice of monic basis of descending degrees $f_1(u), f_2(u), \ldots, f_r(u)$, of X is called the *coordinate map.*

Remark 4.8.14. The coordinate map is not continuous! This is easily seen in an example. Let $r = 2$ and $d = 3$. Consider the 1-parameter family of subspaces

$$X(s) = \mathbb{C}\{u^2 + s, u\}.$$

In this case $\theta(X(s)) = \left(s - u^2, u\right)$. However

$$X_\infty = \lim_{s\to\infty} X(s) = \mathbb{C}\{1, u\}.$$

So $\theta(X_\infty) = (1,1)$, which is clearly not the same as $\lim_{s\to\infty} \theta(X(s)) = (1, u)$.

Essentially the problem in Remark 4.8.14 is the monic basis of descending degrees which we are using to calculate $\theta(X(s))$ no longer has descending degrees in the limit. Whenever we can find a continuous (or holomorphic, or algebraic) family of monic bases of descending degrees the map θ will be continuous (or holomorphic, or algebraic) since taking the Wronskian of a tuple of polynomials is algebraic. An important case in which we can do this is for a a Schubert cell. If $X \in \Omega^\circ(\mu^{\mathsf{c}};\infty)$, we can find a unique basis of the form

$$f_i(u) = u^{d_i} + \sum_{\substack{j=1 \\ d_i - j \notin \mathbf{d}}} a_{ij} u^{d_i - j},$$

where $\mathbf{d} = (d_1, d_2, \ldots, d_r)$. The a_{ij} define algebraic coordinates on $\Omega^\circ(\mu^{\mathsf{c}};\infty)$. Hence θ is algebraic when restricted to any open Schubert cell. In Section 5.6.1 we will prove that the coordinate map is continuous along certain paths in $\mathrm{Gr}(r,d)$ which are allowed to have limit points outside a Schubert cell.

Theorem 4.8.15 ([MTV12, Theorem 5.3])**.** *The image of* $\Omega^\circ(\square^n, \mu^{\mathsf{c}}; z, \infty)$ *under the coordinate map is contained in* $\mathcal{C}(\square^n; z)_\mu$.

Chapter 5

Monodromy actions of the cactus group

In this chapter we prove Theorems A and B. Our first step in proving Theorem A will be to identify the monodromy of Speyer's covering $\vartheta_{\lambda_\bullet,\mu^c}$ (restricted to the real points) with the action of PJ_n on $\mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$. In fact we will do something slightly more general than this. In Section 5.1 we show there is an S_n-action on $\bigsqcup_{\lambda_\bullet} \mathcal{S}(\lambda_\bullet, \mu^c)$ where the $\lambda_\bullet$ ranges over an appropriate set of sequences of partitions, and that the equivariant monodromy of this family is given by the action of the full cactus group J_n on $\bigsqcup_{\lambda_\bullet} \mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$. We then apply Theorem B.1.6 of Harris [Har79] to prove Theorem A. Theorem B is proved by first giving an alternative inductive description of the labelling of points in the fibre of $\Omega^\circ(\square^n, \mu^c)$ by standard μ-tableaux. We are then apply to apply an induction argument to prove Theorem B.

5.1 The S_k-action

Given a permutation $\sigma \in S_k$ and a subset $A \subset [k]$ we use the notation

$$\sigma A = \{\sigma(a) \,|\, a \in A\}\,.$$

In this way σ defines a permutation of the set of three element sets $A \subset [k]$. The permutation σ also induces an isomorphism $C_A \to C_{\sigma A}$ (sending marked points to marked points) and hence an isomorphism $\mathrm{Gr}(d,r)_A \to \mathrm{Gr}(d,r)_{\sigma A}$. In order to keep our notation tidy we will use σ to denote all of these isomorphisms, the context should make it clear which we a referring to. Since the constructions were functorial, diagrams of the type

$$\begin{tikzcd} \mathrm{Gr}(r,d)_A \arrow[rr, "\sigma"] & & \mathrm{Gr}(r,d)_{\sigma A} \\ & \mathrm{Gr}(r,d)_{\mathbb{P}^1} \arrow[ul, "\phi_A"] \arrow[ur, "\phi_{\sigma A}"'] & \end{tikzcd} \tag{5.1.1}$$

commute. Let S_k act on $\overline{M}_{0,k}(\mathbb{C})$ by permuting marked points. The above discussion means we have an action of S_k on the trivial family

$$\overline{M}_{0,k}(\mathbb{C}) \times \prod_{A \in \binom{[k]}{3}} \mathrm{Gr}(d,r)_A.$$

The symmetric group S_k also acts on sequences of k partitions by permuting them in the obvious way.

Proposition 5.1.1. *The variety $\mathcal{G}(d,r)$ is stable under the action of S_k and the variety $\mathcal{S}(\lambda_\bullet)$ is sent isomorphically onto $\mathcal{S}(\sigma \cdot \lambda_\bullet)$. In particular the stabiliser of $\lambda_\bullet$ in S_k acts on $\mathcal{S}(\lambda_\bullet)$.*

Proof. Recall $\mathcal{G}(r,d)$ is defined as the closure of the image of the embedding

$$\begin{aligned} M_{0,k}(\mathbb{C}) \times \mathrm{Gr}(r,d) &\hookrightarrow \overline{M}_{0,k}(\mathbb{C}) \times \textstyle\prod_A \mathrm{Gr}(r,d)_A \\ (C,X) &\longmapsto (C, \phi_A(C,X)). \end{aligned}$$

By the commutativity of (5.1.1), the S_k-action preserves the image of the trivial bundle $M_{0,k}(\mathbb{C}) \times \mathrm{Gr}(r,d)$. Thus the action of S_k also preserves the closure.

Let $A \subseteq [k]$ be a three element set. By Lemma 4.1.7, for any $a \in A$ the isomorphism σ sends $\Omega(\lambda_a, a)_A$ to the Schubert variety $\Omega(\lambda_a, \sigma(a))_{\sigma A}$. This means

$$\begin{aligned} \sigma\mathcal{S}(\lambda_\bullet) &= \sigma\left(\mathcal{G}(r,d) \cap \bigcap_{a\in A} \Omega(\lambda_a, a)_A\right) \\ &= \mathcal{G}(r,d) \cap \bigcap_{a\in A} \Omega(\lambda_a, \sigma(a))_{\sigma A} \\ &= \mathcal{G}(r,d) \cap \bigcap_{a\in A} \Omega(\lambda_{\sigma^{-1}(a)}, a)_A. \end{aligned}$$ □

5.1.1 Acting on the ν labelling

We will need some finer information on what the symmetric group orbits in $\mathcal{S}(\lambda_\bullet)$ look like. This will be helpful when it comes to determining what the S_k-action does to the cylindrical growth diagram indexing a face of $\mathcal{S}(\square^k)(\mathbb{R})$.

Consider the fibre $\mathcal{G}(r,d)(C)$ for some stable curve $C \in \overline{M}_{0,k}(\mathbb{C})$. Using the description of the fibre from Theorem 4.2.5, we have

$$\mathcal{G}(r,d)(C) = \bigcup_{\nu \in \mathtt{N}_C} \prod_i \bigcap_{d \in D_i} \Omega(\nu(C_i,d),d)_{C_i},$$

where $\mathtt{N}_C$ is the set of node labellings for C, C_i the irreducible components of C and D_i the set of nodes on the component C_i. The action of S_k on $\overline{M}_{0,k}(\mathbb{C})$ permutes marked points, thus C and its image σC are the same curve simply with different marked points.

That is, there is an isomorphism $C \to \sigma C$ which we can take to be the identity morphism, which sends the point marked by a to the point marked by $\sigma(a)$. In this way we identify the irreducible components C_i and σC_i and if d is a node in C_i, we also have a node $d \in \sigma C_i$. Using this identification, a node labelling $\nu \in \mathtt{N}_C$ naturally determines a node labelling in $\mathtt{N}_{\sigma C}$, which we also denote by ν.

Lemma 5.1.2. *The S_k action on $\mathcal{G}(d, r)$ preserves the ν-component of the fibre. More precisely, if we fix $\nu \in \mathtt{N}_C$ the image of*

$$\prod_i \bigcap_{d \in D_i} \Omega(\nu(C_i, d), d)_{C_i}$$

under the action of σ is

$$\prod_i \bigcap_{d \in \sigma D_i} \Omega(\nu(\sigma C_i, d), d)_{\sigma C_i}.$$

Proof. By the commutativity of (5.1.1), the Grassmannian $\mathrm{Gr}(r, d)_{C_i}$ is sent isomorphically onto $\mathrm{Gr}(r, d)_{\sigma C_i}$. Lemma 4.1.7 then shows that $\Omega(\nu(C_i, d), d)_{C_i}$ is mapped onto $\Omega(\nu(\sigma C_i, d), d)_{\sigma C_i}$. □

5.2 The S_k-action on $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$

We can now describe how the S_k-action effects the labelling of the fibre by cylindrical growth diagrams.

Proposition 5.2.1. *If Θ is an associahedron in $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$ labelled by (s, γ) then $\sigma\Theta$ is the associahedron labelled by $(\sigma \cdot s, \gamma)$.*

Proof. Let $\hat{\Theta} = \sigma\Theta$. We first restrict to the fundamental case when $\lambda_\bullet = (\square^k)$. The action of S_k on $\overline{M}_{0,k}(\mathbb{R})$ permutes the marked points, hence $\sigma \cdot s$ is the circular ordering labelling the associahedron $\hat{\Theta}$. Recall the cylindrical growth diagram $\hat{\gamma}$ of $\hat{\Theta}$ is determined by considering the ν-labelling of a point on each of its facets. As shown in Lemma 5.1.2 this ν-labelling is preserved, so $\hat{\gamma} = \gamma$.

Now consider the general case for arbitrary $\lambda_\bullet$. We use the notation from Section 4.6.2. Choose a permutation $\tilde{\sigma} \in S_{\tilde{k}}$ such that

$$\Theta \times \prod_{i=1}^{k} \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^C)(\mathbb{R}) \subset \tilde{\Theta}$$

is sent to

$$\sigma \cdot \Theta \times \prod_{i=1}^{k} \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^C)(\mathbb{R}) \subset \tilde{\sigma} \cdot \tilde{\Theta}.$$

Here $\tilde{\Theta}$ is a choice of associahedron in $\mathcal{S}(\Box^{\tilde{k}})(\mathbb{R})$. If $\tilde{\gamma}$ is the cylindrical growth diagram labelling $\tilde{\Theta}$, then the above shows $\tilde{\gamma}$ is also the cylindrical growth diagram labelling $\tilde{\sigma} \cdot \gamma$. But the decgd labelling $\sigma \cdot \Theta$ is by definition the reduction of this cylindrical growth diagram, which by assumption is γ. □

5.2.1 The equivariant monodromy

Let $k = n+1$, fix a basepoint $C \in M_{0,n+1}(\mathbb{R})$ and consider the sequence of partitions $(\lambda_\bullet, \mu^{\mathsf{c}})$ where $\lambda_\bullet = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $|\mu| = |\lambda_\bullet|$ (note that this condition implies $|\lambda_\bullet| + |\mu^{\mathsf{c}}| = r(d-r)$). Let $S_n^{\lambda_\bullet} \subseteq S_n$ be the subgroup fixing $\lambda_\bullet$. Proposition 5.1.1 says we have an action of S_n on the disjoint union

$$\bigsqcup_\sigma \mathcal{S}(\sigma \cdot \lambda_\bullet, \mu^{\mathsf{c}})(\mathbb{R}),$$

where σ ranges over a set of representatives for the cosets $S_n/S_n^{\lambda_\bullet}$. The cactus group J_n acts on this family by equivariant monodromy.

Using Theorem 4.6.5, identify the fibre over C with $\bigsqcup_\sigma \mathtt{decgd}(\sigma \cdot \lambda_\bullet, \mu^{\mathsf{c}})$. We can do this in a number of ways but fix one by choosing, for the associahedron containing C, a representation $s = (s(1), s(2), \dots, s(n+1))$ of the corresponding circular ordering. Let γ be the decgd labelling an associahedron lying over C and let $\hat{\gamma}$ be the decgd labelling the associahedron obtained from γ by crossing the wall corresponding to flipping the marked points $s(p), s(p+1), \dots, s(q)$.

Corollary 5.2.2. *The equivariant monodromy action of the cactus group J_n on the set $\bigsqcup_\sigma \mathtt{decgd}(\sigma \cdot \lambda_\bullet, \mu^{\mathsf{c}})$ is given by $s_{pq} \cdot \gamma = \hat{\gamma}$.*

Proof. Recall from Section 4.5.2, s_{pq} acts by monodromy around the equivariant loop $(\alpha, \hat{s}_{pq})$ where α is a path from C to $\hat{s}_{pq} \cdot C$ passing through the wall which swaps the marked points $s(p), s(p+1), \dots, s(q)$. We lift α to $\tilde{\alpha}$, the unique path in the covering space $\bigsqcup_\sigma \mathcal{S}(\sigma \cdot \lambda_\bullet, \mu^{\mathsf{c}})(\mathbb{R})$ starting at the point over C labelled γ. By Proposition 4.6.6 the point over $\hat{s}_{pq} \cdot C$ at the end of $\tilde{\alpha}$ is labelled $\hat{\gamma}$. Now Proposition 5.2.1 says acting by $\hat{s}_{pq}$ does not change the decgd. Hence $s_{pq} \cdot \gamma = \hat{\gamma}$. □

5.2.2 The fundamental case for μ

We now restrict to the case when $\lambda_\bullet = (\Box^n)$. In this case $S_n^{\lambda_\bullet} = S_n$ so the fibre over C is identified with $\mathtt{decgd}(\Box^n, \mu^{\mathsf{c}})$. For $n = 5$ a decgd of shape $(\Box^n, \mu^{\mathsf{c}})$ will have the form shown in Figure 5.1. Note the partition in position $(1,6)$ is μ (the bottom left corner is in position $(1,1)$). This is demonstrated by the following lemma.

Lemma 5.2.3. *If $\gamma \in \mathtt{decgd}(\Box^n, \mu^{\mathsf{c}})$ then $\gamma_{1(n+1)} = \mu$.*

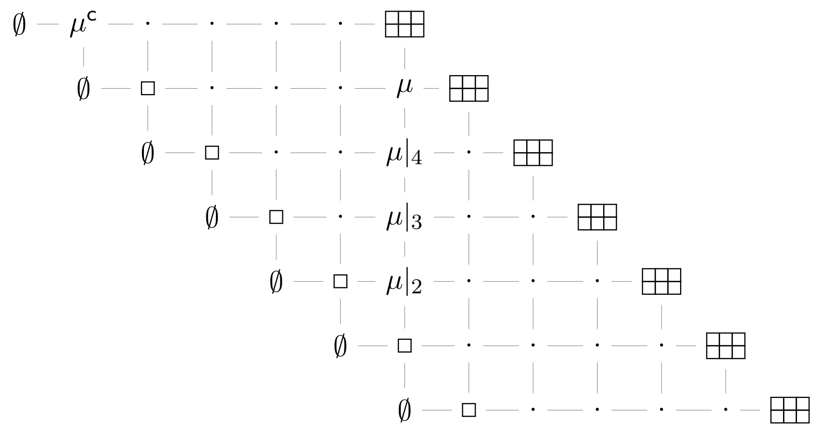


Figure 5.1: A decgd of shape $(\square^5, \mu^c)$

Proof. By definition $\gamma_{(n+1)(n+2)} = \mu^c$. Consider the rectangular subdiagram with corners $(n+1, n+1), (n+1, n+2), (1, n+1)$ and $(1, n+2)$. Let $\gamma_{1(n+1)} = \nu$, our goal is to show $\nu = \mu$.

$$S\left\{\begin{tikzcd} \mu^c \arrow[rr, dash] \arrow[d, dash] & & \Lambda_{r,d} \arrow[d, dash] \\ \emptyset \arrow[rr, dash] & & \nu \end{tikzcd}\right\}T$$

Choose tableaux S and T lifting the dual equivalence classes as shown in the diagram above and extend the rectangular region to a growth diagram (which is unique once S and T are chosen). Since our rectangle is a growth diagram, S must be the rectification of T. Note that $c^{\Lambda_{r,d}}_{\nu\mu^c}$ is the number of $T \in \mathtt{SYT}(\Lambda_{r,d} \backslash \nu)$ slide equivalent to S (see Theorem 4.1.10), since we have produced such a T, it must be the case that $c^{\Lambda_{r,d}}_{\nu\mu^c} > 0$. But $c^{\Lambda_{r,d}}_{\nu\mu^c} = \delta_{\nu\mu}$ (see [Ful97, Section 9.4]) hence $\nu = \mu$. $\square$

We can use Lemma 5.2.3 to give a bijection between $\mathtt{decgd}(\square^n, \mu^c)$ and $\mathtt{SYT}(\mu)$ by choosing a path though $\mathbb{I}$. The standard Young tableaux associated to a decgd γ is described by the growth sequence along the path. Fix the unique path from $(1, 1)$ to $(1, n+2)$ and denote it α. Use α to identify $\mathtt{decgd}(\square^n, \mu^c)$ with $\mathtt{SYT}(\mu)$.

There is an action of the cactus group J_n on $\mathtt{SYT}(\mu)$ by *partial Schützenberger involutions.* This action was studied by Berenstein and Kirillov [BK95]. The partial Schützenberger involution of order q on $T \in \mathtt{SYT}(\mu)$ is defined by applying the Schützenberger involution to the subtableau $T|_q$ and leaving the remaining entries (i.e. those in $T|_{q+1,n}$) unchanged.

Proposition 5.2.4. *The identification of* $\mathtt{SYT}(\mu)$ *and* $\mathtt{decgd}(\square^n, \mu^c)$ *above identifies the action of* J_n *on both sides. More precisely if* T *is obtained using the path* α *from the decgd* γ *then the standard tableaux* $s \cdot T$ *is obtained by taking the path* α *through the decgd* $s \cdot \gamma$*, for all* $s \in J_n$*.*

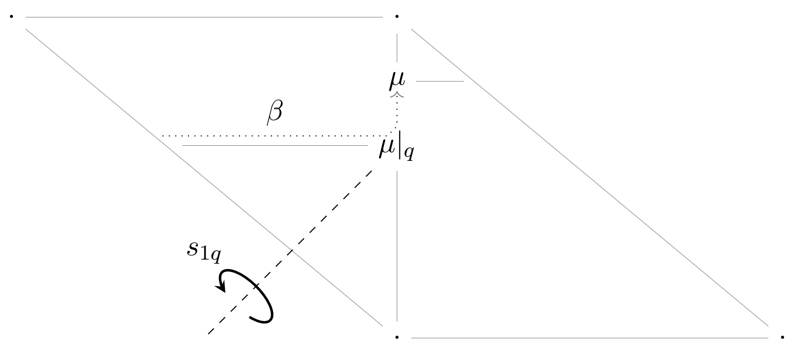


Figure 5.2: The action of s_{1q}

Proof. We only need to show this for $s = s_{1q}$ by Lemma 4.5.1. Let γ be the decgd with T along the path α. Denote the shape of $T|_q$ by $\mu|_q$, so $\gamma_{1(q+1)} = \mu|_q$. Consider the triangle in γ depicted in Figure 5.2. Proposition 4.6.6 says that $s_{1q} \cdot \gamma$ will contain the same triangle, flipped about the axis shown. In particular the tableau obtained along the path α in $s_{1q} \cdot \gamma$ is the same as the tableau obtained along the path β from $(q+1, q+1)$ to $(1, q+1)$ and then to $(1, n+2)$ in γ. By Corollary 4.3.4 this is the partial Schützenberger involution $s_{1q} \cdot T$. □

Corollary 5.2.5. *Identify the fibre in $\mathcal{S}(\square^n, \mu^{\mathsf{c}})(\mathbb{R})$ over C with $\mathtt{SYT}(\mu)$ as described above. The equivariant monodromy action of J_n is given by partial Schützenberger involutions.*

5.3 Monodromy in $\mathcal{S}(\lambda_\bullet, \mu^{\mathsf{c}})$ and crystals

The aim of this section is to show that we may identify the set $\mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$ with the set of dual equivalence growth diagrams of shape $(\lambda_\bullet, \mu^{\mathsf{c}})$ and that this identifies the action of the cactus group on both sides. We start by understanding the action of J_n on $\mathtt{B}^{\otimes n}$. Then we prove results allowing us to understand the actions of of the cactus group J_n on $\mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$ and $\mathtt{decgd}(\lambda_\bullet, \mu^{\mathsf{c}})$ in terms of the action of J_n on $\mathtt{B}^{\otimes n}$.

5.3.1 Crystals and partial Schützenberger involutions

Recall from Section 5.2.2 the action of J_n on $\mathtt{SYT}(\mu)$ by partial Schützenberger involutions.

Proposition 5.3.1. *The bijection $[\mathtt{B}^{\otimes n}]_\mu^{\mathsf{sing}} \longrightarrow \mathtt{SYT}(\mu); w \mapsto \mathtt{Q}(w)$, given by taking the Q-symbol of a word, is equivariant for the action of J_n.*

Proof. Recall from Lemma 4.5.1 that the elements s_{1q} for $1 < q \leq n$ generate J_n. As before we identify $\mathtt{B}^{\otimes n}$ with $\mathtt{words}(n)$. Let $w = b_1 b_2 \cdots b_n \in [\mathtt{B}^{\otimes n}]_\mu^{\mathsf{sing}}$ be a highest weight word and let $Q = \mathtt{Q}(w)$. Denote the subword $b_1 \cdots b_q$ by w_q. Recall the notation w^* from Section 3.2.6. The word $s_{1q} \cdot w$ is by definition

$$\xi(\xi(b_q)\xi(b_{q-1})\cdots\xi(b_1))b_{q+1}\cdots b_n = \xi(w_q^*)b_{q+1}\cdots b_n.$$

By Lemma 3.3.24 the involution ξ does not change the Q-symbol of a word so we have that $\mathtt{Q}(\xi(w_q^*)) = \mathtt{Q}(w_q^*)$ and by Theorem 3.2.20, $\mathtt{Q}(w_q^*) = \mathtt{evac}\,\mathtt{Q}(w_q)$. By considering the definition of the RSK correspondence by the insertion algorithm $\mathtt{Q}(w_q) = Q|_{1,q}$. Thus $\mathtt{Q}(s_{1,q} \cdot w)|_{1,q} = \mathtt{Q}(\xi(w_q^*)) = \mathtt{evac}\ Q|_{1,q}$. The remaining letters in the word $s_{1q} \cdot w$ have not changed, therefore $\mathtt{Q}(s_{1q} \cdot w)|_{q+1,n} = Q|_{q+1,n}$. Thus $s_{1q} \cdot Q = \mathtt{Q}(s_{1q} \cdot w)$. □

5.3.2 Standard $\lambda_\bullet$-tableaux

Let $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a sequence of partitions. Let $\tilde{n} = |\lambda_\bullet|$ and $n_i = |\lambda_i|$. We also define $m_i = n_1 + n_2 + \ldots + n_{i-1}$.

Definition 5.3.2. A *(semi-)standard tableau* of shape $\lambda_\bullet$ is a sequence of tableaux $T_\bullet = (T_1, T_2, \ldots, T_n)$ such that T_i is a (semi-)standard tableau of shape λ_i.

Suppose $w = x_1 x_2 \cdots x_{\tilde{n}} \in \mathtt{words}(\tilde{n})$, define the subwords corresponding to the i^{th} partition, $w_i = x_{m_i+1}x_{m_i+2}\cdots x_{m_{i+1}}$.

Definition 5.3.3. If $\mathtt{P}(w_i)$ has shape λ_i for all i, we say w has $\lambda_\bullet$*-partial P-symbols* $\mathtt{P}(w_\bullet) = (\mathtt{P}(w_1), \mathtt{P}(w_2), \ldots, \mathtt{P}(w_n))$. Similarly, if $\mathtt{Q}(w_i)$ has shape λ_i we say w has $\lambda_\bullet$*-partial Q-symbols* $\mathtt{Q}(w_\bullet) = (\mathtt{Q}(w_1), \mathtt{Q}(w_2), \ldots, \mathtt{Q}(w_n))$. These are semistandard and standard $\lambda_\bullet$-tableaux respectively.

5.3.3 Equivariant embeddings of crystals

In this section we define embeddings $\imath_{T_\bullet}$ of $\mathtt{B}(\lambda_\bullet)^{\mathsf{sing}}_\mu$ into $[\mathtt{B}^{\otimes\tilde{n}}]^{\mathsf{sing}}_\mu$ such that the following diagram commutes.

$$\begin{array}{ccc} [\mathtt{B}(\lambda_\bullet)]^{\mathsf{sing}}_\mu & \xhookrightarrow{\imath_{T_\bullet}} & \left[B^{\otimes\tilde{n}}\right]^{\mathsf{sing}}_\mu \\ \Big\downarrow{\scriptstyle s_{1q}} & & \Big\downarrow{\scriptstyle \bar{s}_{1q}} \\ [\mathtt{B}(\hat{s}_{1q}\cdot\lambda_\bullet)]^{\mathsf{sing}}_\mu & \xhookrightarrow{\imath_{\hat{s}_{1q}\cdot T_\bullet}} & \left[B^{\otimes\tilde{n}}\right]^{\mathsf{sing}}_\mu \end{array} \tag{5.3.1}$$

Here s_{1q} is a generator of J_n, $\hat{s}_{1q}$ its image in S_n and $\bar{s}_{1q} \in J_{\tilde{n}}$ is a particular element we construct now. The idea is that the image of $\bar{s}_{1q}$ in $S_{\tilde{n}}$ acts by preserving the blocks of the first $|\lambda_1|$ letters, the next $|\lambda_2|$ letters and so on, while permuting these n blocks in the same way as $\hat{s}_{1q}$. Let $m_i = \sum_{j=1}^{i-1}|\lambda_j|$ and $m_i^q = \sum_{j=0}^{i-1}\left|\lambda_{\hat{s}_{1q}(j)}\right|$. Denote the generators of $J_{\tilde{n}}$ by $\tilde{s}_{kl}$. Define

$$\bar{s}_{1q} = \left(\prod_{i=1}^{q}\tilde{s}_{(m_i^q+1)m_{i+1}^q}\right)\tilde{s}_{(m_p+1)m_q}.$$

Fix a standard $\lambda_\bullet$-tableau $T_\bullet$. We can use this to obtain a morphisms of crystals $\mathtt{read}_{T_i} : \mathsf{B}(\lambda_\bullet) \hookrightarrow \mathsf{B}^{\otimes|\lambda_i|}$. We can take the tensor product of these embeddings to obtain an embedding

$$\mathtt{read}_{T_\bullet} = \mathtt{read}_{T_1} \otimes \mathtt{read}_{T_2} \otimes \cdots \otimes \mathtt{read}_{T_n} : \mathsf{B}(\lambda_\bullet) \longleftrightarrow \mathsf{B}^{\otimes \tilde{n}}.$$

Since $\mathtt{read}_{T_\bullet}$ is a morphism of crystals, it preserves weights and highest weight elements. Hence we can restrict $\mathtt{read}_{T_\bullet}$ to an injection

$$\imath_{T_\bullet} : \mathsf{B}(\lambda_\bullet)^{\mathsf{sing}}_\mu \longleftrightarrow [\mathsf{B}^{\otimes \tilde{n}}]^{\mathsf{sing}}_\mu.$$

Proposition 5.3.4. *The diagram (5.3.1) commutes.*

Proof. Let $b = b_1 \otimes \cdots \otimes b_n \in [\mathsf{B}(\lambda_\bullet)]^{\mathsf{sing}}_\mu$. By definition $\imath_{\hat{s}_{1q}\cdot T_\bullet} \circ s_{1q}(b)$ has $\hat{s}_{1q} \cdot \lambda_\bullet$-partial Q-symbols $\hat{s}_{1q} \cdot T_\bullet$. Our first job is to show the same is true for $\bar{s}_{1q} \circ \imath_{T_\bullet}(b)$, i.e. $\bar{s}_{1q} \circ \imath_{T_\bullet}(b)$ lies in the same copy of $[\mathsf{B}(\lambda_\bullet)]^{\mathsf{sing}}_\mu$.

By definition $w = \imath_{T_\bullet}(b)$ has $\lambda_\bullet$-partial Q-symbols $T_\bullet$, that is, suppose $w = x_1 x_2 \cdots x_{\tilde{n}}$ then $x_{m_i+1} \cdots x_{m_{i+1}}$ has Q-symbol T_i. We will use the notation $w|_{i,j}$ for the subword $x_i \cdots x_j$. Let $T'_\bullet$ be the $(\hat{s}_{1q} \cdot \lambda_\bullet)$-partial Q-symbols of $\bar{s}_{1q} \cdot w$. If $i > q$ then $(\bar{s}_{1q} \cdot w)\,|_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}} = w|_{m_i+1, m_{i+1}}$, so

$$T'_i = \mathtt{Q} \circ \mathtt{RSK}\left((\bar{s}_{1q} \cdot w)\,|_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}}\right) = \mathtt{Q} \circ \mathtt{RSK}\left(w|_{m_i+1, m_{i+1}}\right) = T_i.$$

Since $\hat{s}_{1q}(i) = i$, we have $T_i = T_{\hat{s}_{1q}(i)}$ for $i > q$. On the other hand, if $i < q$, let $s_{1m_q} \cdot w = y_1 y_2 \cdots y_{\tilde{n}}$. By definition,

$$\begin{aligned} T'_i &= \mathtt{Q} \circ \mathtt{RSK}\left((\bar{s}_{1q} \cdot w)\,|_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}}\right) \\ &= \mathtt{Q} \circ \mathtt{RSK}\left(\left(s_{(m_i^{s_{1q}}+1)m_{i+1}^{s_{1q}}} \cdot y_1 y_2 \cdots y_{\tilde{n}}\right)|_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}}\right), \end{aligned}$$

where we have used the definition of $\bar{s}_{1q}$. Now applying the definition of the cactus group action on words,

$$T'_i = \mathtt{Q} \circ \mathtt{RSK}\left(\xi(y^*_{m_{i+1}^{s_{1q}}} \cdots y^*_{m_i^{s_{1q}}+1})\right).$$

Using the fact that ξ preserves the Q-symbol of a word and applying Theorem 3.2.20 we obtain

$$T'_i = \mathtt{evac} \circ \mathtt{Q} \circ \mathtt{RSK}\left(y_{m_i^{s_{1q}}+1} \cdots y_{m_{i+1}^{s_{1q}}}\right).$$

Now we can apply the rectification property for Q-symbols of subwords from Proposition 3.2.21, so

$$T'_i = \mathtt{evac} \circ \mathtt{Rect}\left(\mathtt{Q} \circ \mathtt{RSK}\,(y_1 \cdots y_{\tilde{n}})\,|_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}}\right)$$

$$= \texttt{evac} \circ \texttt{Rect}\left(\texttt{Q} \circ \texttt{RSK}\left(\xi(x^*_{m_q} \cdots x^*_1) x_{m_q+1} \cdots x_{\tilde{n}}\right)|_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}}\right)$$
$$= \texttt{evac} \circ \texttt{Rect}\left(\texttt{Q} \circ \texttt{RSK}\left(x^*_{m_q} \cdots x^*_1 x_{m_q+1} \cdots x_{\tilde{n}}\right)|_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}}\right),$$

where we have used the definition of $y_1 \dots y_{\tilde{n}}$ and the fact that ξ preserves Q-symbols again. Picking out the correct subword and applying Theorem 3.2.20 gives

$$T'_i = \texttt{evac} \circ \texttt{Q} \circ \texttt{RSK}\left(x^*_{m_{s_{1q}(i)+1}} \cdots x^*_{m_{s_{1q}(i)}+1}\right)$$
$$= \texttt{evac} \circ \texttt{evac} \circ \texttt{Q} \circ \texttt{RSK}\left(x_{m_{s_{1q}(i)}+1} \cdots x_{m_{s_{1q}(i)+1}}\right).$$

Since evac is an involution,

$$T'_i = \texttt{Q} \circ \texttt{RSK}\left(x_{m_{s_{1q}(i)}+1} \cdots x_{m_{s_{1q}(i)+1}}\right)$$
$$= T_{\hat{s}_{1q}(i)}.$$

By Theorem 4.4.3, property (iii) two words are dual equivalent if an only they have the same Q-symbol. Thus the above shows we have divided $\imath_{\hat{s}_{1q} \cdot T_\bullet} \circ s_{1q}(b)$ and $\bar{s}_{1q} \circ \imath_{T_\bullet}(b)$ up into dual equivalent words. Proposition 4.4.2, showed that dual equivalence was a local property, hence $\imath_{\hat{s}_{1q} \cdot T_\bullet} \circ s_{1q}(b)$ and $\bar{s}_{1q} \circ \imath_{T_\bullet}(b)$ are dual equivalent words. Since they are by definition highest weight words they are slide equivalent as well as being dual equivalent and thus are the same word as the intersection of and dual equivalence and slide equivalence class is a single element by Theorem 4.4.3, property ii. □

5.3.4 Equivariant embeddings of decgds

Again let $T_\bullet$ be a standard $\lambda_\bullet$-tableaux. For $\gamma \in \texttt{decgd}(\lambda_\bullet, \mu^c)$ we can lift this to $\texttt{decgd}(\square^{\tilde{n}}, \mu^c)$ by simply choosing a representative for each dual equivalence class along the path from $(1, 1)$ to $(1, n+2)$. We want to do this in a controlled way. Let α_i be the dual equivalence class allocated to the edge $(1, i) - (1, i+1)$. Choose a lift S_i of α_i such that S_i is slide equivalent to T_i. Since the intersection of any slide equivalence class and dual equivalence class is a single tableaux (Theorem 4.4.3), we have a unique choice for S_i. The map $\jmath_{T_\bullet}$ is defined by sending γ to the above described decgd in $\texttt{decgd}(\square^{\tilde{n}}, \mu^c)$.

Proposition 5.3.5. *The diagram,*

$$\begin{tikzcd} \texttt{decgd}(\lambda_\bullet, \mu^c) \arrow[r, hook, "\jmath_{T_\bullet}"] \arrow[d, "s_{1q}"] & \texttt{decgd}(\square^{\tilde{n}}, \mu^c) \arrow[d, "\bar{s}_{1q}"] \\ \texttt{decgd}(\hat{s}_{1q} \cdot \lambda_\bullet, \mu^c) \arrow[r, hook, "\jmath_{\hat{s}_{1q} \cdot T_\bullet}"] & \texttt{decgd}(\square^{\tilde{n}}, \mu^c) \end{tikzcd}, \quad (5.3.2)$$

commutes.

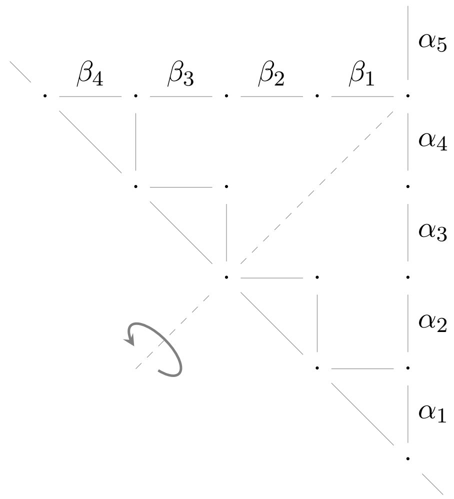


Figure 5.3: The dual equivalence classes α_i and β_i.

Proof. By Corollary 5.2.2, the action of the cactus group on decgd's is given by the rotation of certain triangles. Let $\gamma \in \texttt{decgd}(\lambda_\bullet, \mu^{\mathsf{c}})$. We will first calculate the tableaux defined by the growth along the path from $(1,1)$ to $(1, \tilde{n}+2)$ in $\jmath_{\bar{s}_{1q} \cdot T_\bullet}(s_{1q} \cdot \gamma)$. As depicted in Figure 5.3 (for $q = 4$) let α_i be the dual equivalence class on the edge connecting $(1,i)$ and $(1,i+1)$, for $1 \leq i \leq n$ and β_i the dual equivalence class of of the edge connecting $(q+1,i)$ and $(q+1,i+1)$ for $1 \leq i \leq q$. Furthermore let U_i and V_i be the unique standard tableaux of dual equivalence classes α_i and β_i respectively which are slide equivalent to T_i.

The action of s_{1q} flips the triangle about the axis shown in Figure 5.3 and preserves the partitions and dual equivalence classes along the path from $(1, q+1)$ to $(1, n+2)$ by Proposition 4.6.6. By definition, $\jmath_{\hat{s}_{1q} \cdot T_\bullet}(s_{1q} \cdot \gamma)$ is then constructed by lifting the appropriate dual equivalence classes along the path $(1,1) - (1, \tilde{n}+2)$ to the following tableaux,

$$V_q, V_{q-1}, \cdots V_1, U_{q+1}, \cdots U_n. \tag{5.3.3}$$

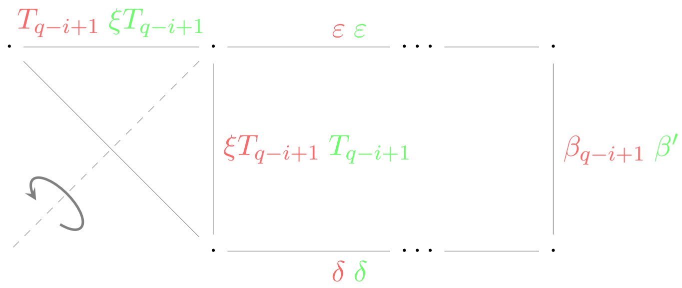
This determines $\jmath_{\hat{s}_{1q} \cdot T_\bullet}(s_{1q} \cdot \gamma)$. Now we make the same calculation for the other side of the commutative diagram.

First apply $\jmath_{T_\bullet}$ to γ, which means lifting the dual equivalence classes along the path $(1,1) - (1, n+2)$ (these are the classes α_i) to U_i. Now apply $\overline{s}_{1q}$, this means flipping a large triangle and several smaller triangles. Figure 5.4 depicts (for $q = 4$) the resulting diagram after flipping *only* the large triangle. We have only marked the dual equivalence classes on the vertical and not the actual tableaux.

Now we flip the small triangles, working right to left. The order we flip does not matter as these elements of the cactus group commute. The first triangle is easy, by Proposition 4.6.6 we preserve all the other small triangles as well as the entire path

$(1, |\lambda_q| + 1) - (1, \tilde{n} + 2)$. We end up with $T_q = V_q$ along the path $(1, 1) - (1, |\lambda_q| + 1)$.

The triangles further to the right take some more thought, we will try and flip the i^{th} triangle. Flipping this triangle preserves all the small triangles to the left and the right as well as everything on the path $(1, 1) - (1, \tilde{n} + 2)$ except for the section between $(1, m_i^{\hat{s}_{1q}} + 1)$ and $(1, m_{i+1}^{\hat{s}_{1q}} + 1)$. Locally we have the picture



where we have marked the dual equivalence classes and tableaux before the flip in red and after the flip in green. In fact $\beta' = \beta_{q-i+1}$. To see this denote by η the dual equivalence class of T_i, this is also the dual equivalence class of ξT_i by Theorem 4.4.3 (i). Thus (δ, β_{q-i+1}) is the shuffle of (η, ε). However (δ, β') is also the shuffle of (η, ε), thus $\beta' = \beta_{q-i+1}$.

What we have shown is that after flipping all of the triangles, (i.e. after applying $\overline{s}_{1q}$ to $\jmath_{T_\bullet}(\gamma)$) the dual equivalence class in the i^{th} position on the path $(1, 1) - (1, m_q + 1)$ is β_{q-i+1} and the tableaux in this position is thus the unique tableaux in β_{q-i+1} slide equivalent to T_{q-i+1}. By assumption this is V_{q-i+1}. Since we never changed anything on the path $(1, m_q + 1) - (1, \tilde{n} + 2)$, the tableaux along the path $(1, 1) - (1, \tilde{n} + 2)$ are

$$V_q, V_{q-1}, \cdots V_1, U_{q+1}, \cdots U_n,$$

which coincides with (5.3.3). Hence $\jmath_{\hat{s}_{1q} \cdot T_\bullet}(s_{1q} \cdot \gamma) = \overline{s}_{1q} \cdot \jmath_{T_\bullet}(\gamma)$. $\square$

5.3.5 Crystals and dual equivalence cylindrical growth diagrams

Recall that Proposition 5.3.1 says the bijection $\mathtt{Q} : [\mathsf{B}^{\otimes \tilde{n}}]^{\mathsf{sing}}_\mu \longrightarrow \mathtt{SYT}(\mu)$, defined by taking the Q-symbol, is $J_{\tilde{n}}$-equivariant. We can use a standard μ-tableau S to construct a decgd $\alpha(S)$ of shape $(\square^{\tilde{n}}, \mu^c)$ by requiring the growth in $\alpha(S)$ along the path from $(1, 1)$ to $(1, n+1)$ defines S. By Proposition 5.2.4 $\alpha : \mathtt{SYT}(\mu) \longrightarrow \mathtt{decgd}(\square^{\tilde{n}}, \mu^c)$ is a $J_{\tilde{n}}$-equivariant bijection. Combining these results we obtain the following proposition.

Proposition 5.3.6. *The map $\alpha \circ \mathtt{Q}$ provides a $J_{\tilde{n}}$ equivariant bijection between $[\mathsf{B}^{\otimes \tilde{n}}]^{\mathsf{sing}}_\mu$ and $\mathtt{decgd}(\square^{\tilde{n}}, \mu^c)$.*

Lemma 5.3.7. *Let $T_\bullet$ be a standard $\lambda_\bullet$-tableau. The bijection $\alpha \circ \mathtt{Q}$ identifies the images of $\imath_{T_\bullet}$ and $\jmath_{T_\bullet}$.*

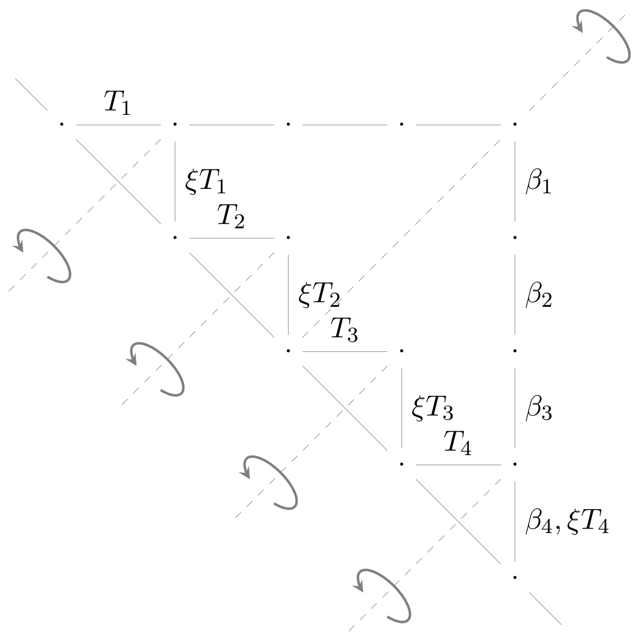


Figure 5.4: $\overline{s}_{1q}$ acting on a decgd

Proof. Let $b \in \mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$, let $w = x_1 x_2 \cdots x_{\tilde{n}} = \imath_{T_\bullet}(b)$ and let $S = \mathtt{Q} \circ \imath_{T_\bullet}(b)$. In order to prove the Lemma, we need to show that the rectification of $S_i = S|_{m_i+1, m_{i+1}}$ is the tableau T_i. By construction $\imath_{T_\bullet}(b)$ has $\lambda_\bullet$-partial Q-symbol $T_\bullet$. By Proposition 3.2.21, the rectification of S_i is the Q symbol of $x_{m_i+1} x_{m_i+2} \cdots x_{m_{i+1}}$, which is T_i by assumption. $\square$

As in Section 5.2.1, let $S_n^{\lambda_\bullet}$ be the stabiliser of $\lambda_\bullet$ in S_n. The cactus group acts on the sets

$$\bigsqcup_\sigma \mathtt{B}(\sigma \cdot \lambda_\bullet)_\mu^{\mathsf{sing}} \quad \text{and} \quad \bigsqcup_\sigma \mathtt{decgd}(\sigma \cdot \lambda_\bullet, \mu^{\mathsf{c}}), \tag{5.3.4}$$

where σ ranges over representatives of the cosets $S_n / S_n^{\lambda_\bullet}$. We define a bijection $\mathtt{Q}_{\lambda_\bullet,\mu}$ between the sets (5.3.4) by requiring that $\mathtt{Q}_{\lambda_\bullet,\mu}$ restricted to $\mathtt{B}(\sigma \cdot \lambda_\bullet)_\mu^{\mathsf{sing}}$ is given by $\jmath_{T_\bullet}^{-1} \circ \alpha \circ \mathtt{Q} \circ \imath_{T_\bullet}$. When $\lambda_\bullet = (\square^n)$ we set $\mathtt{Q}_\mu = \mathtt{Q}_{\lambda_\bullet,\mu}$.

Theorem 5.3.8. *The map $\mathtt{Q}_{\lambda_\bullet,\mu}$ does not depend on $T_\bullet$ and is a J_n-equivariant bijection.*

Proof. For the fundamental case when $\lambda_\bullet = (\square^n)$, both $\imath_{T_\bullet}$ and $\jmath_{T_\bullet}$ are identity maps and the statement reduces to Proposition 5.3.6. For the general case, the map $\mathtt{Q}_{\lambda_\bullet,\mu}$ depends, by construction, only on the dual equivalence classes of $T_\bullet$. However, $\mathtt{SYT}(\lambda_i)$ consists of a single dual equivalence class by Theorem 4.4.3 and hence $\mathtt{Q}_{\lambda_\bullet,\mu}$ does not depend on $T_\bullet$. For any $2 \le q \le n$, we have a diagram,

$$\begin{tikzcd}
\left[\mathtt{B}(\lambda_\bullet)\right]^{\mathsf{sing}}_{\mu} \arrow[r, hook, "\imath_{T_\bullet}"] \arrow[d, "s_{1q}"] & \left[\mathtt{B}^{\otimes \tilde{n}}\right]^{\mathsf{sing}}_{\mu} \arrow[r, "\alpha\circ\mathtt{Q}''"] \arrow[d, "\tilde{s}_{1q}"] & \mathtt{decgd}(\square^{\tilde{n}}, \mu) \arrow[d, "\tilde{s}_{1q}"] & \mathtt{decgd}(\lambda_\bullet, \mu) \arrow[l, hook', "\jmath_{T_\bullet}"'] \arrow[d, "s_{1q}"] \\
\left[\mathtt{B}(\hat{s}_{1q}\cdot\lambda_\bullet)\right]^{\mathsf{sing}}_{\mu} \arrow[r, hook, "\imath_{\hat{s}_{1q}\cdot T_\bullet}"] & \left[\mathtt{B}^{\otimes \tilde{n}}\right]^{\mathsf{sing}}_{\mu} \arrow[r, "\alpha\circ\mathtt{Q}''"] & \mathtt{decgd}(\square^{\tilde{n}}, \mu) & \mathtt{decgd}(\hat{s}_{1q}\cdot\lambda_\bullet, \mu). \arrow[l, hook', "\jmath_{\hat{s}_{1q}\cdot T_\bullet}"']
\end{tikzcd}$$

By Proposition 5.3.6, the centre square of this diagram commutes and by Propositions 5.3.4 and 5.3.5, the left and right squares also commute. This implies commutativity of the entire diagram. Since J_n is generated by the elements s_{1q} (Lemma 4.5.1) this demonstrates $\mathtt{Q}_{\lambda_\bullet,\mu}$ is J_n-equivariant. □

5.4 The proof of Theorem A

We have now built up enough structure and are ready to prove Theorem A. We restate the theorem here.

Theorem A. *There exists a homomorphism $PJ_n \to \mathrm{Gal}(\pi_{\lambda_\bullet,\mu})$ from the pure cactus group to the Galois group of $\pi_{\lambda_\bullet,\mu}$ and a bijection*

$$\mathcal{A}(\lambda_\bullet, z)_\mu \longrightarrow \mathtt{B}(\lambda_\bullet)^{\mathsf{sing}}_\mu,$$

equivariant for the induced action of PJ_n.

Proof. The four varieties we have been investigating and their relationship is summarised by the following diagram.

$$\begin{tikzcd}
\mathcal{A}(\lambda_\bullet)_\mu \arrow[r, "\theta"] \arrow[d, "\pi_{\lambda_\bullet,\mu}"] & \Omega(\lambda_\bullet, \mu^{\mathsf{c}}) \arrow[r, hook, "\iota"] \arrow[d, "\vartheta^\circ_{\lambda_\bullet,\mu^{\mathsf{c}}}"] & \mathcal{S}(\lambda_\bullet, \mu^{\mathsf{c}}) \arrow[d, "\vartheta_{\lambda_\bullet,\mu^{\mathsf{c}}}"] & \mathcal{S}(\lambda_\bullet, \mu^{\mathsf{c}})(\mathbb{R}) \arrow[l, hook'] \arrow[d, "\vartheta_{\lambda_\bullet,\mu^{\mathsf{c}}}|_{\mathbb{R}}"] \\
M_{0,n+1}(\mathbb{C}) \arrow[r, "="] & M_{0,n+1}(\mathbb{C}) \arrow[r, hook] & \overline{M}_{0,n+1}(\mathbb{C}) & \overline{M}_{0,n+1}(\mathbb{R}) \arrow[l, hook']
\end{tikzcd} \tag{5.4.1}$$

The map $\vartheta_{\lambda_\bullet,\mu^{\mathsf{c}}}|_{\mathbb{R}}$ is the restriction of $\vartheta_{\lambda_\bullet,\mu^{\mathsf{c}}}$ to the real points and by Theorem 4.2.3 is a topological covering of $\overline{M}_{0,n+1}(\mathbb{R})$. For the sake of notational clarity we will set $\pi = \pi_{\lambda_\bullet,\mu}$, $\vartheta^\circ = \vartheta^\circ_{\lambda_\bullet,\mu^{\mathsf{c}}}$, and $\vartheta = \vartheta_{\lambda_\bullet,\mu^{\mathsf{c}}}$.

The first claim of Theorem A is that, for a generic point $z \in M_{0,n+1}(\mathbb{C})$, there is a homomorphism $PJ_n \to \mathrm{Gal}(\pi; z)$. First, assume that z is real, that is $z \in M_{0,n+1}(\mathbb{R})$. Let $M_{\mathbb{R}} \subset S_{\vartheta^{-1}(z)}$ be the monodromy group of the unramified covering $\vartheta|_{\mathbb{R}}$. Since $PJ_n = \pi_1(\overline{M}_{0,n+1}(\mathbb{R}); z)$, we have a surjective homomorphism $PJ_n \to M_{\mathbb{R}}$.

Choose a dense open subset $z \in U \subset \overline{M}_{0,n+1}(\mathbb{C})$ over which ϑ is unramified. We can choose U so that is contains $\overline{M}_{0,n+1}(\mathbb{R})$ by Theorem 4.2.3. Denote by $M_U(\vartheta; z)$ the associated monodromy group. The inclusion of the real points $\mathcal{S}(\lambda_\bullet, \mu^{\mathsf{c}})(\mathbb{R})$ induces an inclusion $M_{\mathbb{R}} \hookrightarrow M_U(\vartheta; z)$. Harris' Theorem B.1.6 implies that $M_U(\vartheta; z) = \mathrm{Gal}(\vartheta; z)$.

The group $\mathrm{Gal}(\pi; z)$ is a subgroup of $S_{\pi^{-1}(z)}$. The morphism $\iota \circ \theta$ identifies the sets $\pi^{-1}(z)$ and $\eta^{-1}(z)$. With this identification fixed, $\mathrm{Gal}(\pi; z) = \mathrm{Gal}(\vartheta; z)$ by Proposition B.1.4. Hence we have a homomorphism from PJ_n onto the subgroup $M_{\mathbb{R}} \subseteq \mathrm{Gal}(\pi; z)$.

In the case $z \notin \overline{M}_{0,n+1}(\mathbb{R})$, fix a point $w \in \overline{M}_{0,n+1}(\mathbb{R})$ and a path from z to w. This choice fixes a identification of the fibres $\pi^{-1}(z)$ and $\pi^{-1}(w)$ and thus an identification of the Galois groups $\mathrm{Gal}(\pi; z)$ and $\mathrm{Gal}(\pi; w)$. We use this identification to produce a homomorphism $PJ_n \to \mathrm{Gal}(\pi; z)$.

The second claim of Theorem A is that for a real point $z \in M_{0,n+1}(\mathbb{R})$ there exists a bijection of sets $\mathcal{A}(\lambda_\bullet; z)_\mu \to \mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$ equivariant for the action of PJ_n. The isomorphism θ identifies $\mathcal{A}(\lambda_\bullet; z)_\mu$ with $\Omega(\lambda_\bullet, \mu^{\mathsf{c}}; z, \infty)$. By definition this identification is equivariant for the action of PJ_n. By Theorem 4.6.5 $\Omega(\lambda_\bullet, \mu^{\mathsf{c}}; z, \infty)$ can be identified with $\mathtt{decgd}(\lambda_\bullet, \mu^{\mathsf{c}})$. Now we may use Theorem 5.3.8 to find a bijection to $\mathtt{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$ which is equivariant with respect to J_n (and thus PJ_n). $\square$

5.5 Speyer's labelling

In this section we recall Speyer's labelling of the set $\Omega(\square^n, \mu^{\mathsf{c}}; z, \infty)$ by standard μ-tableaux for real z. We will give an alternative description of this labelling which is what we will ultimately use to prove Theorem B.

For a three element set $A \subset [n+1]$ recall from Section 4.2.1 the definition of the Grassmanian $\mathrm{Gr}(r,d)_A$. Given an n-tuple of distinct complex numbers, recall as well the isomorphism $\phi_A : M_{0,n+1}(\mathbb{C}) \times \mathbb{P}^1 \longrightarrow M_{0,n+1}(\mathbb{C}) \times C_A$. Recall that we have an embedding

$$\iota_\mu : \Omega(\square^n, \mu^{\mathsf{c}}) \longrightarrow \mathcal{S}(\square^n, \mu^{\mathsf{c}}) \subset \overline{M}_{0,n+1}(\mathbb{C}) \times \prod_{T \subset [n+1]} \mathrm{Gr}(r,d)_T$$

given by $\iota_\mu(z, X) = (\phi_A(z, X))_A$.

Let $z = (z_1, z_2, \ldots, z_n)$ be a tuple of distinct real numbers with the property that $z_1 < z_2 < \ldots < z_n$ and let C be the stable curve with one component and marked points $(z_1, z_2, \ldots, z_n, \infty)$. Recall Speyer's labelling of the fibre $\mathcal{S}(\square^n, \mu^{\mathsf{c}})(C)$ by dual equivalence cylindrical growth diagrams of shape $(\square^n, \mu^{\mathsf{c}})$ from Section 4.6. In Section 5.2.2 we described how this can be interpreted as a labelling by $\mathtt{SYT}(\mu)$.

We say $X \in \Omega(\square^n, \mu^{\mathsf{c}}; z, \infty)$ is labelled by $S \in \mathtt{SYT}(\mu)$ if $(\phi_A(z, X))_A$ is labelled by S in the fibre $\mathcal{S}(\square^n, \mu^{\mathsf{c}})(C)$. We denote the unique point in $\Omega(\square^n, \mu^{\mathsf{c}}; z, \infty)$ determined by S by X_S.

5.5.1 An alternative labelling

In this section we will describe an alternative method of labelling points in the fibres of $\Omega(\square^n, \mu^{\mathsf{c}})$ by standard μ-tableaux. Fix an n-tuple of distinct real numbers $z \in X_n(\mathbb{R})$

such that $z_1 < z_2 < \ldots < z_n$ and fix a subspace $X \in \Omega(\square^n, \mu^{\mathsf{c}}; z, \infty)$. We will associate to it a standard μ-tableaux. We will do this by induction on n.

For $n = 1$, there is a single point in $\Omega(\square, \square^{\mathsf{c}}; z, \infty)$ and a single $\square$-tableaux so the labelling is completely determined. For $n > 1$ let $P_n \subset M_{0,n+1}(\mathbb{R})$ be the subvariety where we fix the first $n-1$ marked points at $z_1, \ldots, z_{n-1}$, the last marked point at ∞ and let the n^{th} marked point vary. Thus, as real varieties, $P_n \simeq \mathbb{R} - \{z_1, z_2, \ldots, z_{n-1}\}$. Consider the restriction $\Omega(\square^n, \mu^{\mathsf{c}})|_{P_n}$ and the natural projection $p : \Omega(\square^n, \mu^{\mathsf{c}})|_{P_n} \longrightarrow \mathrm{Gr}(r, d)$. Let $X(s)$ be a local section of $\Omega(\square^n, \mu^{\mathsf{c}})|_{P_n}$ (a family over P_n) such that $X(z_n) = X$. Let

$$X_\infty = \lim_{s \to \infty} pX(s) \in \mathrm{Gr}(r, d).$$

Lemma 5.5.1. *The limit point X_∞ exists and is contained in $\Omega^\circ(\lambda^{\mathsf{c}}, \infty)$ for some partition $\lambda \subset \mu$ such that $|\lambda| + 1 = |\mu|$. In particular*

$$X_\infty \in \Omega(\square^{n-1}, \lambda^{\mathsf{c}}; z_1, \ldots, z_{n-1}, \infty).$$

Proof. Since the Grassmanian is projective, X_∞ must exist and since a Schubert variety is a union of smaller Schubert cells,

$$\Omega(\mu^{\mathsf{c}}; \infty) = \bigsqcup_{\nu \supseteq \mu^{\mathsf{c}}} \Omega^\circ(\nu; \infty),$$

we must have that $X_\infty \in \Omega^\circ(\lambda^{\mathsf{c}}, \infty)$ for some partition λ such that $\lambda^{\mathsf{c}} \supseteq \mu^{\mathsf{c}}$, or equivalently, such that $\lambda \subseteq \mu$.

Since each of the varieties $\Omega(\square; z_i)$ is closed, X_∞ must lie in the intersection $\Omega(\square^{n-1}, \lambda^{\mathsf{c}}; z_1, \ldots, z_{n-1}, \infty)$. However this is empty unless $|\lambda| \geq n-1$. Hence either $\lambda = \mu$ or λ is obtained by removing a single box from μ. To decide whether $|\lambda| = n$ or $n-1$ we use the Wronskian. By [MTV09a, Lemma 4.2]

$$\mathrm{Wr}(X(s))(u) = (u - s) \prod_{a=1}^{n-1} (u - z_a).$$

By continuity, $\mathrm{Wr}(X_\infty) = \prod_{a=1}^{n-1}(u - z_a)$. It is straightforward to see from the definition that if $Y \in \Omega^\circ(\nu^{\mathsf{c}}; \infty)$ then $\deg \mathrm{Wr}(Y) = |\nu|$. We have shown that $\mathrm{Wr}(X_\infty) = n - 1$ so therefore $|\lambda| = n - 1$. $\square$

Summarising, $X_\infty \in \Omega(\square^{n-1}, \lambda^{\mathsf{c}}; z_1, \ldots, z_{n-1}, \infty)$ and $|\lambda| = n-1$. Thus by induction we can assign a standard λ-tableau to X_∞. Let this tableau be T'. Let T be the unique, standard μ-tableaux which is T' upon restriction to $\lambda \subset \mu$. We let T be the standard μ-tableau labelling X.

5.5.2 Levinson result

In this section we briefly outline a result of Levinson [Lev15] that will be used in Section 5.5.3. Recall that $\mathcal{S}(\lambda_\bullet, \Box)$ is a subvariety of $\overline{M}_{0,n+1} \times \prod_A \mathrm{Gr}(r,d)_A$, where A ranges over three element subsets of $[n+1]$. Let π be the projection onto $\prod_{A \subset [n]} \mathrm{Gr}(r,d)_A$. That is π projects onto those Grassmanians for subsets A which do not contain $n+1$.

Suppose $|\lambda_\bullet| = r(d-r) - 1$ and let $c_{n+1} : \overline{M}_{0,n+1} \longrightarrow \overline{M}_{0,n}$ be the contraction map at the point marked by $n+1$ (see Section 2.3.2). The morphism c_{n+1} allows us to think of $\mathcal{S}(\lambda_\bullet, \Box)$ as a family over $\overline{M}_{0,n}(\mathbb{C})$ (rather than $\overline{M}_{0,n+1}(\mathbb{C})$).

Theorem 5.5.2 ([Lev15, Theorem 2.8]). *The map π produces an isomorphism onto $\mathcal{S}(\lambda_\bullet)$ and we have the following commutative diagram,*

$$\begin{array}{ccc} \mathcal{S}(\lambda_\bullet, \Box) & \xrightarrow{\quad \pi \quad} & \mathcal{S}(\lambda_\bullet) \\ \downarrow & & \downarrow \\ \overline{M}_{0,n+1}(\mathbb{C}) & \xrightarrow{c_{n+1}} & \overline{M}_{0,n}(\mathbb{C}). \end{array}$$

Thus π is an isomorphism of families over $\overline{M}_{0,n}(\mathbb{C})$.

5.5.3 The labellings agree

Note that a priori there is no reason two points of $\Omega(\Box^n, \mu^c; z, \infty)$ do not share the same labelling tableau as defined in Section 5.5.1. We show in this section that the labelling does associate a unique point to every tableaux, and in fact, this labelling coincides with Speyer's labelling. For clarity of exposition, the bulk of the proof is organised into a series of lemmas below. Let $X = X_T$ and let $X(s) \in \mathrm{Gr}(r,d)$ be as in Section 5.5.1.

Lemma 5.5.3. *Let $\lambda = \mathrm{sh}(T|_{n-1})$, then $X_\infty = \lim_{s\to\infty} X(s) \in \Omega^\circ(\lambda^c; \infty)$.*

Proof. Let ι_μ be the inclusion $\Omega(\Box^n, \mu^c) \hookrightarrow \mathcal{S}(\Box^n, \mu^c)$. That is, for (C, X) a point of $\Omega(\Box^n, \mu^c) \subseteq M_{0,n+1} \times \mathrm{Gr}(r,d)$,

$$\iota_\mu(C,E) = (\phi_A(C,E))_{A \subset [n+1]}.$$

If p is the projection $\Omega(\Box^n, \mu^c) \to \mathrm{Gr}(r,d)$ and $p_{\{1,2,3\}}$ is the composition of the projection $\mathcal{S}(\Box^n, \mu^c) \to \mathrm{Gr}(r,d)_{\{1,2,3\}}$ with

$$\phi^{-1}_{\{1,2,3\}} : \mathrm{Gr}(r,d)_{\{1,2,3\}} \longrightarrow \mathrm{Gr}(r,d),$$

$$\begin{tikzcd}
 & & \mathrm{Gr}(r,d) & & \\
\Omega(\square^n,\mu^{\mathsf{c}}) \arrow[urr, "p"] \arrow[r, hook, "\iota_\mu"] \arrow[d] & \mathcal{S}(\square^n,\mu^{\mathsf{c}}) \arrow[r, "\pi"] \arrow[d] & \mathcal{S}(\square^{n-1},\mu^{\mathsf{c}}) \arrow[u, "p_{\{1,2,3\}}"'] \arrow[d] & \mathcal{S}(\square^{n-1},\lambda^{\mathsf{c}}) \arrow[l, hook', "\zeta"'] \arrow[d] & \Omega(\square^n,\lambda^{\mathsf{c}}) \arrow[l, hook', "\iota_\lambda"'] \arrow[ull, "p'"'] \arrow[d] \\
M_{0,n+1} \arrow[r, hook] & \overline{M}_{0,n+1} \arrow[r] & \overline{M}_{0,n} & \overline{M}_{0,n} \arrow[l] & M_{0,n} \arrow[l, hook']
\end{tikzcd}$$

Figure 5.5: The relationship between various projections

then we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Gr}(r,d) \\
\Omega(\square^n,\mu^{\mathsf{c}}) \arrow[ur, "p"] \arrow[r, hook, "\iota_\mu"] \arrow[d] & \mathcal{S}(\square^n,\mu^{\mathsf{c}}) \arrow[u, "p_{\{1,2,3\}}"'] \arrow[d] \\
M_{0,n+1} \arrow[r, hook] & \overline{M}_{0,n+1}.
\end{tikzcd}$$

Let C be the stable curve with marked points z and $C(s)$ the family of stable curves with marked points $(z_1,\dots,z_{n-1},s,\infty)$, so $C = C(z_n)$. We have

$$\begin{aligned}
X_\infty &= \lim_{s\to\infty} p(C(s),X(s)) \\
&= \lim_{s\to\infty} p_{\{1,2,3\}}\iota_\mu(C(s),X(s)) \\
&= p_{\{1,2,3\}}Y,
\end{aligned}$$

where $Y = \lim_{s\to\infty}\iota_\mu(C(s),X(s))$.

The point Y lies over the stable curve $\lim_{s\to\infty} C(s)$, which has two components, C_1 with marked points $z_1,z_2,\dots,z_{n-1}$ and a node at ∞, and C_2 with marked points at 1 and ∞ and a node at 0. Thus by Theorem 4.2.5,

$$Y \in \Omega(\square^{n-1},\lambda^{\mathsf{c}};z_1,\dots,z_{n-1},\infty)_{C_1} \times \Omega(\lambda,\square,\mu^{\mathsf{c}};0,1,\infty)_{C_2}.$$

The partition λ appearing in the node labelling must be $\mathrm{sh}(T|_{n-1})$ since we assumed $X = X_T$. Now $p_{\{1,2,3\}}$ is simply projection onto the first factor and is an isomorphism onto $\mathrm{Gr}(r,d)$ so $X_\infty = p_{\{1,2,3\}}Y \in \Omega^\circ(\lambda^{\mathsf{c}};\infty)$. $\square$

We would now like to calculate the Speyer labelling of the point X_∞ in $\mathcal{S}(\square^{n-1},\lambda^{\mathsf{c}})$. However we only have information about the labelling of the points $X(s)$ in $\mathcal{S}(\square^n,\mu^{\mathsf{c}})$. To relate these two covering spaces we will use Theorem 5.5.2. With this theorem we produce the large commutative diagram in Figure 5.5.

In Figure 5.5 ι_μ is the inclusion $\Omega(\square^n, \mu^{\mathsf{c}}) \hookrightarrow \mathcal{S}(\square^n, \mu^{\mathsf{c}})$ described above. The inclusion ι_λ is defined similarly. The morphism π is the isomorphism appearing in Theorem 5.5.2 and the inclusion ζ is induced by the inclusion of $\Omega(\lambda^{\mathsf{c}}; \infty)$ into $\Omega(\mu^{\mathsf{c}}; \infty)$.

Lemma 5.5.4. *Let $Y = \lim_{s\to\infty} \iota_\mu(C(s), X(s))$, and let C_∞ be the stable curve with marked points $(z_1, z_2, \ldots, z_{n-1}, \infty)$ (the component C_1 as in the proof of Lemma 5.5.3). Then*

$$Y = \pi^{-1}\zeta\iota_\lambda(C_\infty, X_\infty).$$

Proof. We will show $p_{\{1,2,3\}}\pi Y = p_{\{1,2,3\}}\zeta\iota_\lambda(C_\infty, X_\infty)$. Since $p_{\{1,2,3\}}$ is injective on fibres, this is enough to prove the Lemma. This amounts to tracing X_∞ around the diagram. By commutativity of the diagram,

$$p_{\{1,2,3\}}\zeta\iota_\lambda(C_\infty, X_\infty) = p'(C_\infty, X_\infty) = X_\infty.$$

Now

$$\begin{aligned} p_{\{1,2,3\}}\pi Y &= p_{\{1,2,3\}}\pi \lim_{s\to\infty} \iota_\mu(C(s), X(s)) \\ &= \lim_{s\to\infty} p_{\{1,2,3\}}\pi\iota_\mu(C(s), X(s)) \\ &= \lim_{s\to\infty} p(C(s), X(s)) = X_\infty. \end{aligned}$$

□

Lemma 5.5.5. *Let Θ be the associahedron in $\mathcal{S}(\square^{|\nu|}, \nu^{\mathsf{c}})$ labelled by $S \in \mathtt{SYT}(\nu)$. For generic $E \in \Theta_{1q}$*

$$p_{\{1,2,3\}}E \in \Omega^\circ(\tau^{\mathsf{c}}; \infty),$$

where $\tau = \mathrm{sh}(S|_q)$.

Proof. This is a direct application of the Theorem 4.2.5, which says if E is generic then

$$E \in \Omega(\square^q, \tau^{\mathsf{c}}; u_1, \ldots, u_q, \infty)_{C_1} \times \Omega(\tau, \square^{|\nu|-q}, \nu^{\mathsf{c}}; 0, u_{q+1}, \ldots, u_{|\nu|}, \infty)_{C_2}.$$

Then $p_{\{1,2,3\}}$ is projection onto the first factor. □

Remark 5.5.6. The proof of Lemma 5.5.5 shows in fact we can make the stronger assumption that E is a generic point of $\Theta_{1p} \cap \Theta_{1q}$ as long as $p \geq q$.

Lemma 5.5.7. *Let Θ be the $(n-2)$-associahedron in $\mathcal{S}(\square^n, \mu^{\mathsf{c}})$ labelled by T and let $\tilde{\Theta}$ be the $(n-3)$-associahedron in $\mathcal{S}(\square^{n-1}, \lambda^{\mathsf{c}})$ containing $\iota_\lambda(X_\infty, C_\infty)$. Then $\pi^{-1}\zeta(\tilde{\Theta}) = \Theta_{1n}$.*

Proof. Since the maps downstairs in Figure 5.5 are all cell maps, the maps upstairs must also be cell maps. Hence $\pi^{-1}\zeta(\tilde{\Theta})$ must be Θ'_{ij}, the face of some $(n-2)$-associahedron Θ' in $\mathcal{S}(\square^n, \mu^{\mathsf{c}})$. Thus Θ'_{ij} must contain the point $\pi^{-1}\zeta\iota_\lambda(C_\infty, X_\infty)$. By Lemma 5.5.4

$$\pi^{-1}\zeta\iota_\lambda(C_\infty, X_\infty) = Y = \lim_{s\to\infty} \iota_\mu(C(s), X(s)).$$

We know $\iota_\mu(C(s), X(s)) \in \Theta$ so $Y \in \Theta_{1n}$. Hence $\Theta'_{ij} = \Theta_{1n}$. □

Lemma 5.5.8. $X_\infty = X_{T|_{n-1}}$.

Proof. To show the equality we must show the point X_∞ is labelled by the tableau $T|_{n-1}$. That means we must show, for each $2 < q < n$ and a generic point $E \in \tilde{\Theta}_{1q}$, that $p_{\{1,2,3\}}E \in \Omega^\circ(\tau^{\mathsf{c}}; \infty)$ for $\tau = \mathrm{sh}(T|_q)$. By commutativity of Figure 5.5

$$p_{\{1,2,3\}}E = p_{\{1,2,3\}}\pi^{-1}\zeta(E).$$

Lemma 5.5.7 tells us $\pi^{-1}\zeta(\tilde{\Theta}_{1q}) = \Theta_{1q} \cap \Theta_{1n}$. Since $X = X_T \in \Theta$ (which means Θ is the $(n-2)$-associahedron labelled by T) and $\pi^{-1}\zeta(E)$ is generic, we must have that $p_{\{1,2,3\}}\pi^{-1}\zeta(E) \in \Omega^\circ(\tau^{\mathsf{c}}, \infty)$. □

Proposition 5.5.9. *The point $X_T \in \Omega(\square^n, \mu^{\mathsf{c}}; z, \infty)$ is labelled by T according to the process described in Section 5.5.1.*

Proof. According to Lemma 5.5.3 $X_\infty = \lim_{s\to\infty} X(s) \in \Omega^\circ(\lambda^{\mathsf{c}}; \infty)$. Now Lemma 5.5.8 tells us $X_\infty = X_{T|_{n-1}}$. This means the tableaux labelling $X = X_T$ is a tableaux of shape μ whose restriction to $n-1$ is $T|_{n-1}$. However the unique tableaux satisfying these properties is T. □

5.6 The proof of Theorem B

In this section we prove Theorem B. First we present a result which demonstrates the coordinate map is continuous along certain kinds of paths in $\mathrm{Gr}(r, d)$. We then show the alternative labelling of $\Omega(\square^n, \mu^{\mathsf{c}}; z, \infty)$ by standard μ-tableaux, presented in Section 5.5.1 is compatible with the coordinate map and Marcus' labelling of critical points from Section 4.8.5. We use this result to prove Theorem B.

5.6.1 Partial continuity of the coordinate map

We will need a little more information about when the coordinate map is continuous. Let μ be a partition and let $\lambda \subseteq \mu$ be a partition with one less box, that is $|\lambda| = |\mu| - 1$. Denote by e the row of μ from which we need to remove a box to obtain λ.

Let $d_i = \mu_i + r - 1$ and $d'_i = \lambda_i + r - 1$. We denote the respective decreasing sequences by $\mathbf{d} = (d_1, d_2, \ldots, d_r)$ and $\mathbf{d}' = (d'_1, d'_2, \ldots, d'_r)$. Recall that $X \in \Omega^\circ(\mu^{\mathsf{c}}; \infty)$ (respectively $X \in \Omega^\circ(\lambda^{\mathsf{c}}; \infty)$) if and only if there exists a basis $f_1, f_2, \ldots, f_r$ of X such that $\deg f_i = d_i$ (respectively $\deg f_i = d'_i$). In particular if $f \in X$ then $\deg f \in \mathbf{d}$ (respectively $\deg f \in \mathbf{d}'$). Since we removed a single box from μ in row e to obtain λ we have

$$d'_i = \begin{cases} d_i & \text{if } i \neq e \\ d_e - 1 & \text{if } i = e. \end{cases}$$

Fix $X \in \Omega^\circ(\mu^{\mathsf{c}};\infty)$. Let $X(s) \in \Omega^\circ(\mu^{\mathsf{c}};\infty)$ be a continuous one-parameter family over $\mathbb{R}$ of subspaces. We thus have a unique basis

$$f_i(u;s) = u^{d_i} + \sum_{\substack{j=1 \\ d_i-j\notin\mathbf{d}}} a_{ij}(s)u^{d_i-j},$$

for $X(s)$, for each $s\in\mathbb{R}$. The $a_{ij} : \mathbb{R} \longrightarrow \mathbb{C}$ are continuous functions.

Lemma 5.6.1. *The limit point X_∞ of this family is contained in $\Omega^\circ(\lambda^{\mathsf{c}},\infty)$ if and only if*

$$\lim_{s\to\infty} |a_{ij}(s)| < \infty \quad \textit{for } i \neq e \tag{5.6.1}$$

and

$$\lim_{s\to\infty} |a_{e1}(s)| = \infty, \tag{5.6.2}$$

$$\lim_{s\to\infty} \left|\frac{a_{ej}(s)}{a_{e1}(s)}\right| < \infty. \tag{5.6.3}$$

Proof. If properties (5.6.1), (5.6.2) and (5.6.3) hold then in the limit, the basis $f_1,\ldots,f_r$ is a sequence of r polynomials which have descending degrees $d'_1 > d'_2 > \ldots > d'_r$. Hence $X_\infty \in \Omega^\circ(\lambda^{\mathsf{c}};\infty)$.

In the other direction, if (5.6.1) fails to hold, then there exists an $i \neq e$ and a j such that $\lim_{s\to\infty}|a_{ij}(s)| = \infty$. There are two cases. First if $i < e$, then choose j such that $a_{ij}(s)$ has the fastest growth (for fixed i). Thus the polynomial

$$\lim_{s\to\infty} a_{ij}(s)^{-1} f_i(u;s) \in X_\infty$$

has degree $d_i - j$. But $d_i - j \notin \mathbf{d}$ and $d_i - j < d_e - 1$ thus $d_i - j \notin \mathbf{d}'$ and so we must have $X_\infty \notin \Omega^\circ(\lambda^{\mathsf{c}};\infty)$.

Now for the second case, assume there exists an i such that $i > e$ and such that $\lim_{s\to\infty}|a_{ij}(s)| = \infty$. For any $f \in X_\infty$ there exist functions $\alpha_i : \mathbb{R}\longrightarrow\mathbb{C}$ such that

$$f = \lim_{u\to\infty} \alpha_1(s)f_1(u;s) + \alpha_2(s)f_2(u;s) + \ldots + \alpha_r(s)f_r(u;s).$$

If $X_\infty \in \Omega^\circ(\lambda^{\mathsf{c}};\infty)$ then we can choose f so that $\deg f = d_i$. In order for this to be true we first need $\lim_{u\to\infty}\alpha_k(s) = 0$ for $k > i$ (since $\lim_{u\to\infty} f_k(u;s)$ exists in this case and has degree $d_k > d_i$). By assumption

$$\deg \lim_{u\to\infty} \alpha_i(s)f_i(u;s) < d_i$$

and for $k < i$ we also have

$$\deg \lim_{u\to\infty} \alpha_k(s)f_k(u;s) < d_i.$$

Thus we have a contradiction and $X_\infty \notin \Omega^\circ(\lambda^c;\infty)$.

Finally if conditions (5.6.2) and (5.6.3) do not hold then let j be minimal with respect to the condition $a_{ej}(s)$ has the largest order of growth. By assumption $j > 1$. Then the degree of $\lim_{u\to\infty} a_{ej}(s)^{-1} f_e(u;s)$ is $d_e - j \notin \mathbf{d}'$. Hence $X_\infty \notin \Omega^\circ(\lambda^c;\infty)$. □

Using this lemma we can prove the continuity of the coordinate map θ (see Section 4.8.6 for the definition) along certain kinds of paths. Heuristically, the paths along which θ is continuous are those in $\mathrm{Gr}(r,d)$ which remain inside a Schubert cell and if they pass into another Schubert cell, do so only in a way in which the partition labelling the Schubert cell is a single box smaller than the partition labelling the original Schubert cell. Let $X, X(s)$ and X_∞ be as above.

Proposition 5.6.2. *If* $X_\infty \in \Omega^\circ(\lambda^c;\infty)$ *then*

$$\theta(X_\infty) = \theta\left(\lim_{u\to\infty} X(s)\right) = \lim_{u\to\infty} \theta(X(s)).$$

Proof. Since $X_\infty \in \Omega^\circ(\lambda^c;\infty)$ by Lemma 5.6.1 we have conditions (5.6.1), (5.6.2) and (5.6.3) and thus have a monic basis of descending degrees

$$f_i^\infty(u) = \lim_{u\to\infty} f_i(u;s) = u^{d_i} + \sum_{\substack{j=1\\ d_i - j\notin \mathbf{d}}} a_{ij}^\infty u^{d_i - j}, \quad \text{for } i \neq e,$$
$$f_e^\infty(u) = \lim_{u\to\infty} a_{e1}(s)^{-1} f_e(u;s) = u^{d_e - 1} + \sum_{\substack{j=2\\ d_i - j\notin \mathbf{d}}} b_j^\infty u^{d_i - j}.$$

Here $a_{ij}^\infty = \lim_{u\to\infty} a_{ij}(s)$ and $b_j^\infty = \lim_{u\to\infty} a_{ej}(s)/a_{e1}(s)$ which exist by (5.6.1) and (5.6.3).

Let $X^a(s) = \mathbb{C}\{f_a, \dots, f_r\}$ and $X_\infty^a(s) = \mathbb{C}\{f_a^\infty, \dots, f_r^\infty\}$. We can use these spaces to calculate the Wronskian. That is

$$\theta(X(s)) = \left(\mathrm{Wr}(X^1(s)), \dots, \mathrm{Wr}(X^r(s))\right)$$

and

$$\theta(X_\infty) = \left(\mathrm{Wr}(X_\infty^1(s)), \dots, \mathrm{Wr}(X_\infty^r(s))\right)$$

Since $\lim_{s\to\infty} X^a(s) = X_\infty^a(s)$ and since the Wronskian is continuous

$$\begin{aligned}\lim_{u\to\infty} \theta(X(s)) &= \lim_{s\to\infty} \left(\mathrm{Wr}(X^1(s)), \dots, \mathrm{Wr}(X^r(s))\right)\\ &= \left(\mathrm{Wr}(\lim_{s\to\infty} X^1(s)), \dots, \mathrm{Wr}(\lim_{s\to\infty} X^r(s))\right)\\ &= \left(\mathrm{Wr}(X_\infty^1(s)), \dots, \mathrm{Wr}(X_\infty^r(s))\right)\\ &= \theta(X_\infty).\end{aligned}$$

□

5.6.2 The main theorem

In this section we state and prove Theorem B. First we show that the coordinate map θ preserves the labelling. Let $z \in \mathbb{R}^n$ be an n-tuple of distinct real numbers such that $z_1 < z_2 < \cdots < z_n$ and let C be the stable curve with a single component and marked points at $(z_1, z_2, \ldots, z_n, \infty)$.

Theorem 5.6.3. *If T is a standard μ-tableau, the point $X_T \in \Omega(\square^n, \mu^{\mathsf{c}}; z, \infty)$ labelled by T is sent to t_T by the coordinate map. That is, $\theta(X_T) = t_T$.*

Proof. We prove this theorem by induction on n. In the case $n = 1$, Both $\Omega(\square, \square^C)_z$ and $\tilde{\mathcal{C}}(\square)_z$ consist of a single point both of which are labelled by the unique $\square$-tableaux.

For $n > 1$ let $X = X_T \in \Omega(\square^n, \mu^{\mathsf{c}}; z)$ be the point in labelled by $T \in \mathtt{SYT}(\mu)$. As above, let $X(s) \in \Omega(\square^n, \mu^{\mathsf{c}}; z_1, \ldots, z_{n-1}, u, \infty)$ be the unique family of points passing though $X = X_T$. By Lemma 5.5.3, the limit $X_\infty = \lim_{s\to\infty} X(s)$ is contained in $\Omega^\circ(\lambda^{\mathsf{c}}; \infty)$ where $\lambda = \mathrm{sh}(T|_{n-1})$. We also know by Lemma 5.5.8 that X_∞ is the point labelled by $T|_{n-1}$.

By the induction hypothesis $\theta(X_\infty) = t_{T|_{n-1}}$ where $t_{T|_{n-1}}$ is the critical point labelled by $T|_{n-1}$. In particular, by Proposition 5.6.2

$$\lim_{s\to\infty} \theta(X(s)) = \theta(X_\infty) = t_{T|_{n-1}}.$$

However, by Theorem 4.8.8 and by the definition of Marcus' labelling there is a unique family of critical points with this property, namely the family passing through t_T. Hence we have that $\theta(X_T) = t_T$. $\square$

We now have enough information to prove Theorem B which we restate here for convenience.

Theorem B. *For $z = (z_1, z_2, \ldots, z_n)$ an n-tuple of distinct real numbers such that $z_1 < z_2 < \ldots < z_n$, the bijection $\mathbb{X}_\mu(z) : \mathcal{A}(z)_\mu \longrightarrow \mathtt{SYT}(\mu)$ is given by $\mathbb{X}_\mu(z)(\chi) = S$, where S is the unique tableau with*

$$c_S(a) = \lim_{z\to\infty} \chi_S(z_a H_a(z)).$$

Proof. Let $z \in \mathbb{R}^n$ be as above. We can summarise what we have learnt in a diagram in Figure 5.6. In this diagram ω is the universal weight function described in Theorem 4.8.2, κ is the MTV isomorphism from Section 4.7, and θ is the coordinate map from Section 4.8.6. Recall from Section 5.3.5 we have a pair of bijections $\mathtt{Q} : [B^{\otimes n}]^{\mathsf{sing}}_\mu \longrightarrow \mathtt{SYT}(\mu)$ and $\alpha : \mathtt{SYT}(\mu) \longrightarrow \mathtt{decgd}(\square^n, \mu^{\mathsf{c}})$. By Theorem 4.6.5 we also have a bijection $\beta : \mathtt{decgd}(\square^n, \mu^{\mathsf{c}}) \longrightarrow \Omega^\circ(\square^n, \mu^{\mathsf{c}}; z, \infty)$ which provides the labelling of points. In Figure 5.6, η is the composition $\beta \circ \alpha \circ \mathtt{Q}$ and Σ is the bijection from Theorem A and is by definition $\eta^{-1} \circ \kappa$.

$$
\begin{tikzcd}
\mathcal{A}(z)_\mu \arrow[r, "\Sigma"] \arrow[dr, "\kappa"] & {[\mathsf{B}^{\otimes n}]^{\mathsf{sing}}_\mu} \arrow[d, "\eta"] \\
\mathcal{C}(\square^n; z)_\mu \arrow[u, "\omega"] & \Omega^\circ(\square^n, \mu^{\mathsf{c}}; z, \infty) \arrow[l, "\theta"]
\end{tikzcd}
$$

Figure 5.6: The various sets labelled by $\mathtt{SYT}(\mu)$

By [MTV12, Corollary 8.7], $\omega \circ \theta = \kappa^{-1}$, thus the diagram in Figure 5.6 is commutative (the upper right triangle commutes by definition). Since η preserves the labelling by definition we will show that if $\chi_T \in \mathcal{A}(z)_\mu$, considered as a functional $\chi_T : A(z)_\mu \longrightarrow \mathbb{C}$, is the point labelled by $T \in \mathtt{SYT}(\mu)$ then $\kappa(\chi_T) = X_T$. By Theorem 5.6.3, we know that $\theta(X_T) = t_T \in \mathcal{C}(\square^n; z)_\mu$. Let $\chi = \omega(z, t_T)$, again thought of as a functional $\chi : A(x)_\mu \longrightarrow \mathbb{C}$ then by Theorem 4.8.2, (v),

$$\chi(z_a H_a(z)) = z_a \frac{\partial S}{\partial z_a}(z, t_T),$$

so by Theorem 4.8.10,

$$\chi(z_a H_a(z)) = c_T(a) + O(z_a^{-1}).$$

This means by Proposition 2.4.7, $\chi = \chi_T$. $\square$

Appendices

Appendix A

A Gaudin subalgebra with non-simple spectrum

In Section 2.4.4 it was asserted that the Gaudin algebra at the point of $\overline{M}_{0,7}(\mathbb{C})$ labelled by the bracketing (((12)3)((45)6)) does not have simple spectrum on $S(\mu)$ when $\mu = (3, 2, 1)$. Specifically, the operators

$$\begin{aligned}
L_1 &= (1, 2),\\
L_2 &= (1, 3) + (2, 3)\\
L_3 &= (1, 4) + (1, 5) + (1, 6) + (2, 4) + (2, 5) + (2, 6) + (3, 4) + (3, 5) + (3, 6),\\
L_4 &= (4, 5),\\
L_5 &= (4, 6) + (5, 6).
\end{aligned}$$

span this algebra (actually its image in $\mathbb{C}S_6$). In this appendix we collect the data which demonstrates this. Specifically in Section A.1 we provide Magma [BCP97] code which computes the matrix representations of the operators L_i, their eigenvalues and eigenvectors. In Section A.2 extra operators commuting with the L_i are produced which split the two dimensional eigenspaces.

A.1 The non-simple spectrum

The following Magma code outputs the joint eigenspaces of the operators L_1, L_2, L_3, L_4, L_5. The output gives eight, 1-dimensional eigenspaces and 4, 2-dimensional eigenspaces.

```
G := Sym(6);

# Given a list of permutations a, coefficients and a partition,
```

```
# return the corresponding matrix for the irreducible representation
# corresponding to the partition.
L := function(a,coef,part)
  LOp := 0;
  for perm in a do
    LOp +:= SymmetricRepresentation(part, perm);
  end for;
  return ChangeRing(LOp,Rationals());
end function;

# The partition we are interested in
lam := [3,2,1];

# Define the operators
L1 := L( [G!(1,2)],
         [1],
         lam );
L2 := L( [G!(1,3), G!(2,3)],
         [1,1],
         lam );
L3 := L( [G!(1,4), G!(1,5), G!(1,6),
          G!(2,4), G!(2,5), G!(2,6),
          G!(3,4), G!(3,5), G!(3,6)],
         [1,1,1,1,1,1,1,1,1],
         lam );
L4 := L( [G!(4,5)],
         [1],
         lam );
L5 := L( [G!(4,6), G!(5,6)],
         [1,1],
         lam );

# Print joint eigenspaces
CommonEigenspaces( [ L1, L2, L3, L4, L5 ] );
```

The matrices produced above are 16×16 integral matrices which are constructed for the basis of polytabloids of $S(3,2,1)$ as described in [JK81]. We list them below.

$$L_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1
\end{pmatrix}$$

$$L_2 = \begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2
\end{pmatrix}$$

$$L_3 = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 3 & -1 & -3 & 1 & 0 & -2 & -1 & 2 & 1 & -3 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -2 & -1 & 1 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & -3 & 1 & 2 & 1 & -3
\end{pmatrix}$$

$$L_4 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 1
\end{pmatrix}$$

$$L_5 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

The joint eigenspaces of the operators are summarised in Table A.1. All data is relative to the basis of polytabloids of $S(3, 2, 1)$. As discussed in Section 2.4.4, the Gaudin subalgebra at the point $(((12)3)((45)6))$ conjecturally corresponds to the Gelfand-Zetlin subalgebra of the inductive sequence

$$S_1 \subset S_2 \subset S_3 \subset S_3 \times S_2 \subset S_3 \times S_3 \subset S_6.$$

Thus the two dimensional eigenspaces correspond to the fact that the branching graph for this inductive sequence, has four paths along which there is a double edge.

A.2 Higher Hamiltonians with simple spectrum

We saw in Section A.1, that the limits of the Gaudin Hamiltonians do not necessarily have simple spectrum. In this section we write down explicit formulas for higher Hamiltonians in $\mathbb{C}S_n$ which we show have simple spectrum in the particular limiting case considered above.

A.2.1 The higher Hamiltonians

Fix an n-tuple of distinct complex numbers $z = (z_1, z_2, \dots z_n)$. For any ordered set of k integers $\{r_1, \dots, r_k\} \subset [n]$ we have an embedding of $S_k \hookrightarrow S_n$ by sending $i \mapsto r_i$. Denote the image of $\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\sigma \in \mathbb{C}S_k$ in $\mathbb{C}S_n$ under this embedding by $A_{r_1,\dots,r_k}$.

Dim	Spectrum	Basis vectors
2	$-1, 1, 0, -1, 1$	$(1, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0)$ $(0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 1, 0, -1, 0, 0)$
2	$-1, 1, 0, 1, -1$	$(2, 0, 0, 0, 0, -4, 3, 0, 3, 0, 0, -3, 0, -3, 0, 0)$ $(0, 2, 0, 2, 0, 0, -1, 0, -1, 0, 0, -1, 0, -1, 0, 0)$
1	$-1, 1, -3, 1, 2$	$(0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0)$
1	$-1, 1, 3, -1, -2$	$(0, 1, 0 - 1, 0, 0 - 1, 0, 1, 0, 0, 1, 0 - 1, 0, 0)$
1	$-1, -2, 3, 1, -1$	$(1, 0, 0, 0, 0 - 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
1	$-1, -2, 3, -1, 1$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
1	$1, 2, -3, 1, -1$	$(-3, 8, 2, -4, 2, 6, -10, 2, 2, 2, -6, 8, -4, -4, -4, 12)$
1	$1, 2, -3, -1, 1$	$(3, -4, 2, 4, -2, 0, 2, 2, -2, -2, 6, 0, 0, 0, 0, 0)$
2	$1, -1, 0, -1, 1$	$(3, -1, 2, 1, -2, 0, 0, 0, 0, 0, 0, 1, -2, -1, 2, 0)$ $(0, 0, 0, 0, 0, 0, 1, -2, -1, 2, 0, 1, -2, -1, 2, 0)$
2	$1, -1, 0, 1, -1$	$(2, 0, 0, 0, 0, -4, 1, -2, 1, -2, 0, -1, 2, -1, 2, 0)$ $(0, 2, -4, 2, -4, 0, -1, 2, -1, 2, 0, -1, 2, -1, 2, 0)$
1	$1, -1, -3, 1, 2$	$(0, 1 - 2, 1 - 2, 0, 1 - 2, 1 - 2, 0, 1 - 2, 1 - 2, 0)$
1	$1, -1, 3, -1, -2$	$(0, 1 - 2 - 1, 2, 0 - 1, 2, 1 - 2, 0, 1 - 2 - 1, 2, 0)$

Table A.1: The joint eigenvalues and eigenspaces of L_1, L_2, L_3, L_4, L_5

For $1 \leq k \leq n$ the following polynomials in $\mathbb{C}S_n[u]$ are defined in [MTV13, Section 2],

$$\Phi_k(z;u) = \sum_{1 \leq r_1 < \ldots < r_k \leq n} \left(A_{r_1,\ldots,r_k} \prod_{\substack{a=1 \\ a \notin \{r_1,\ldots,r_k\}}}^{n} (u - z_a) \right) = \sum_{l=0}^{n-k} \Phi_{k,l}(z) u^{n-k-l},$$

for some $\Phi_{k,l}(z) \in \mathbb{C}S_n$. In [MTV13] it is proved that these operators generate the Bethe algebra on $V^{\otimes n}$. Recall the notation used in Section 2.2.6.

Proposition A.2.1 ([MTV13, Proposition 2.4 and Theorem 3.2])**.** *The subalgebra $A^S(z) \subset \mathbb{C}S_n$ generated by the operators $\Phi_{k,l}(z)$ for $1 \leq k \leq n$ and $0 \leq l \leq n-k$ is commutative and its image $\rho(A^S(z))$ coincides with the Bethe algebra on*

$$V(z_1) \otimes V(z_2) \otimes \cdots \otimes V(z_n).$$

Let $B_{r_1,\ldots,r_k} = 1 - A_{r_1,\ldots,r_k}$, so in particular $B_{i,j} = (i,j)$. For $0 \leq a < n$ define the following operators

$$X_a^{k+1}(z) = \sum_{\substack{b_1 < \ldots < b_k \\ b_i \neq a}} \frac{B_{a,b_1,\ldots,b_k}}{\prod_{s=1}^k (z_a - z_{b_s})}.$$

In particular $X_a^2(z) = H_a^S(z)$.

Proposition A.2.2. *The operators $X_a^{k+1}(z)$ commute and are elements of the algebra $A^S(z) \subset \mathbb{C}S_n$. More explicitly, we can express $\Phi_{k+1}(z;u)$ as*

$$\Phi_{k+1}(z;u) = \sum_{a=1}^{n} \left(\sum_{\substack{b_1<\ldots<b_k \\ b_i \neq a}} \frac{1}{\prod_{s=1}^{k}(z_a - z_{b_s})} - X_a^{k+1}(z) \right) \prod_{b\neq a}(u - z_b). \tag{A.2.1}$$

Proof. First we prove (A.2.1) by a direct calculation,

$$\begin{aligned}
&\sum_{a=1}^{n} \left(\sum_{\substack{b_1<\ldots<b_k \\ b_i \neq a}} \frac{1}{\prod_{s=1}^{k}(z_a - z_{b_s})} - X_a^{k}(z) \right) \prod_{b\neq a}(u - z_b) \\
&= \sum_{a=1}^{n} \left(\sum_{\substack{b_1<\ldots<b_k \\ b_i \neq a}} \frac{1 - B_{a,b_1,\ldots,b_k}}{\prod_{s=1}^{k}(z_a - z_{b_s})} \right) \prod_{b\neq a}(u - z_b) \\
&= \sum_{a=1}^{n} \sum_{\substack{b_1<\ldots<b_k \\ b_i \neq a}} \left(\frac{A_{a,b_1,\ldots,b_k}}{\prod_{s=1}^{k}(z_a - z_{b_s})} \prod_{i=1}^{k}(u - z_{b_i}) \prod_{b\notin\{a,b_1,\ldots,b_k\}} (u - z_b) \right) \\
&= \sum_{1\leq r_0<r_1<\ldots<r_k\leq n} \left(\sum_{j=0}^{n} \prod_{s=1}^{k} \frac{(u - z_{b_i})}{(z_a - z_{b_s})} \right) A_{a,b_1,\ldots,b_k} \prod_{b\notin\{a,b_1,\ldots,b_k\}} (u - z_b) \\
&= \Phi_{k+1}(z;u).
\end{aligned}$$

The last equality comes from recognising

$$\sum_{j=0}^{n} \prod_{s=1}^{k} \frac{(u - z_{b_i})}{(z_a - z_{b_s})}$$

as the Lagrange interpolating polynomial for the points $(z_{b_i}, 1)$. Since this must be the polynomial with least degree passing through these points it must be the constant polynomial 1.

The identity (A.2.1) lets us write the coefficients $\Phi_{k,l}(z)$ in terms of the $X_a^k(z)$ by expanding and computing the coefficients of u,

$$\Phi_{k,l}(z) = \sum_{a=1}^{n} \left(Y_a^k(z) \sum_{I \in \binom{[n]\setminus a}{k+l}} z_{i_1} \cdots z_{i_{k+l}} \right), \tag{A.2.2}$$

where

$$Y_a^k(z) = \sum_{\substack{b_1<\ldots<b_{k-1} \\ b_i \neq a}} \frac{1}{\prod_{s=1}^{k}(z_a - z_{b_s})} - X_a^k(z),$$

and $\binom{S}{m}$ is the set of subsets of S of size m. We take $I = \{i_1, i_2, \ldots\}$.

Consider A.2.2) for $l > n-k$ as a system of n linear equations in variables $Y_a^k(z)$. The determinant is $\prod_{i<j}(z_i - z_j)$ which is nonzero since the z_i are distinct. Thus we can solve this system and the $Y_a^k(z)$ are elements of $A^S(z)$ and since it is unital algebra so are the $X_a^k(z)$. The operators commute by Proposition A.2.1. $\square$

A.2.2 Limits of the higher Hamiltonians

The algebra $A^S(z) \subset \mathbb{C}S_n$ is a large commutative algebra containing the Gaudin Hamiltonians. In fact, this algebra is generated by the Gaudin Hamiltonians (see[MTV10]) but this need not be true in the limit as the parameters z_a collide with one another (the property of having simple spectrum is an open property). In Section A.1 we saw the Hamiltonians $H_a^S(z)$ do not have simple spectrum in the limit at the point $(((12)3)((45)6)) \in \overline{M}_{0,7}(\mathbb{C})$. The higher Hamiltonians $X_a^k(z)$ provide reasonable candidates for operators, whose limit will decompose the two dimensional eigenspaces appearing in Section A.1.

A.2.3 Degenerate cubic Hamiltonians

If we want to define a Bethe algebra at this degenerate point it should have simple spectrum, that is, it should be maximally commutative. It should also include the Gaudin Hamiltonians. To find the extra operators we need, we can use the definition of the Bethe algebra at the point $z = (0, t^5, t^4, t, t^3 + t, t^2 + t)$ and take limits of cubic, and higher degree Hamiltonians.

First we shall try the cubic Hamiltonians and it will turn out that this is enough. We take the same limit as in Section A.1 as t goes to zero, and again multiplying through by appropriate powers of t we have the following list,

$$\begin{aligned} t^9 X_1^3(z) = {} & B_{123} + t^3 B_{124} + \frac{t^3 B_{125}}{t^2+1} + \frac{t^3 B_{126}}{t+1} + t^5 B_{134} + \frac{t^4 B_{135}}{t^2+1} + \frac{t^4 B_{136}}{t+1} \\ & + \frac{t^7 B_{145}}{t^2+1} + \frac{t^7 B_{146}}{t+1} + \frac{t^7 B_{156}}{t^3+t^2+t+1} \end{aligned}$$

$$\begin{aligned} t^9 X_2^3(z) = {} & \frac{B_{123}}{t-1} + \frac{t^3 B_{124}}{t^4-1} + \frac{t^3 B_{125}}{t^4-t^2-1} + \frac{t^3 B_{126}}{t^4-t-1} + \frac{t^4 B_{234}}{t^5-t^4-t+1} \\ & + \frac{t^4 B_{235}}{t^8-t^4-t^3+t^2-t+1} + \frac{t^4 B_{236}}{t^5-t^4-t^2+1} + \frac{t^7 B_{245}}{t^8-t^6-2t^4+t^2+1} \\ & + \frac{t^7 B_{246}}{t^8-t^5-2t^4+t+1} + \frac{t^7 B_{256}}{t^8-t^6-t^5-2t^4+t^3+t^2+t+1} \end{aligned}$$

$$
\begin{aligned}
t^8 X_3^3(z) = {} & \frac{B_{123}}{1-t} + \frac{t^3 B_{134}}{t^3-1} + \frac{t^3 B_{135}}{t^3-t^2-1} + \frac{t^3 B_{136}}{t^3-t-1} + \frac{t^3 B_{234}}{-t^4+t^3+t-1} \\
& + \frac{t^3 B_{235}}{-t^4+2t^3-t^2+t-1} + \frac{t^3 B_{236}}{-t^4+t^3+t^2-1} + \frac{t^6 B_{345}}{t^6-t^5-2t^3+t^2+1} \\
& + \frac{t^6 B_{346}}{t^6-t^4-2t^3+t+1} + \frac{t^6 B_{356}}{t^6-t^5-t^4-t^3+t^2+t+1}
\end{aligned}
$$

$$
\begin{aligned}
t^5 X_4^3(z) = {} & \frac{t^3 B_{124}}{-t^4+1} + \frac{t^3 B_{134}}{-t^3+1} - tB_{145} - t^2 B_{146} + \frac{t^3 B_{234}}{t^7-t^4-t^3+1} + \frac{tB_{245}}{t^4-1} \\
& + \frac{t^2 B_{246}}{t^4-1} + \frac{tB_{345}}{t^3-1} + \frac{t^2 B_{346}}{t^3-1} + B_{456}
\end{aligned}
$$

$$
\begin{aligned}
t^5 X_5^3(z) = {} & \frac{t^3 B_{125}}{-t^6+2t^2+1} + \frac{t^3 B_{135}}{-t^5+t^4-t^3+2t+1} + \frac{tB_{145}}{t^1+1} + \frac{t^2 B_{156}}{t^3-t^2+t-1} \\
& + \frac{t^3 B_{235}}{t^7-t^6-t^5-t^3+2t^2+1} + \frac{tB_{245}}{-t^4+t^2+1} \\
& + \frac{t^2 B_{256}}{-t^5+t^4+t^3-t^2+t-1} + \frac{tB_{345}}{-t^3+t^2+1} \\
& + \frac{t^2 B_{356}}{-t^4+2t^3-t^2+t-1} + \frac{B_{456}}{t-1}
\end{aligned}
$$

$$
\begin{aligned}
t^4 X_6^3(z) = {} & \frac{t^2 B_{126}}{-t^5-t^4+t^2+2t+1} + \frac{t^2 B_{136}}{-t^4-t^3+t^2+2t+1} + \frac{tB_{156}}{-t^2+1} \\
& + \frac{t^2 B_{236}}{t^7-t^5-2t^4-t^3+t^2+2t+1} + \frac{tB_{246}}{-t^4+t+1} + \frac{tB_{256}}{t^5-t^4-t^2+1} \\
& + \frac{tB_{346}}{-t^3+t+1} + \frac{tB_{356}}{t^4-t^3-t^2+1} + \frac{B_{456}}{-t+1}.
\end{aligned}
$$

Taking the limit of these operators as t goes to zero we obtain two linearly independent operators, $K_1 = B_{123}$ and $K_4 = B_{456}$. Taking the limit of $-t^9 X_2^3(z) - t^8 X_3^3(z)$ and $t^5 X_5^3(z) + t^4 X_6^3(z)$ we obtain the operators

$$
\begin{aligned}
K_2 &= B_{124} + B_{125} + B_{126} + B_{134} + B_{135} + B_{136} + B_{234} + B_{235} + B_{236}, \text{ and} \\
K_3 &= B_{145} + B_{245} + B_{345} + B_{146} + B_{156} + B_{246} + B_{256} + B_{346} + B_{356},
\end{aligned}
$$

respectively. Using the following Magma code we obtain matrix representations for K_1, K_2, K_3 and K_4 and calculate the simultaneous eigenspaces which we also summarise below. This shows that together the L_i and K_i have simple spectrum and generate what deserves to be called a Bethe algebra for the point $z = ((12)3)((45)6) \in \overline{M}_{0,7}$.

```
# Define B-operators for each subset of {1..6}.
B := function(part,r)
```

```
A := 0;
H := {g : g in G | forall{i: i in {1..6} diff r | i^g eq i } };
for sigma in H do
    A +:= Sign(sigma)*SymmetricRepresentation(part, sigma);
end for;
return ChangeRing(ScalarMatrix(Nrows(A),1)-A,Rationals());
end function;

# Define the operators we are interested in.
K1 := B(lam,{1,2,3});
K2 := B(lam,{1,2,4}) + B(lam,{1,2,5}) + B(lam,{1,2,6}) +
      B(lam,{1,3,4}) + B(lam,{1,3,5}) + B(lam,{1,3,6}) +
      B(lam,{2,3,4}) + B(lam,{2,3,5}) + B(lam,{2,3,6});
K3 := B(lam,{1,4,5}) + B(lam,{2,4,5}) + B(lam,{3,4,5}) +
      B(lam,{1,4,6}) + B(lam,{1,5,6}) + B(lam,{2,4,6}) +
      B(lam,{2,5,6}) + B(lam,{3,4,6}) + B(lam,{3,5,6});
K4 := B(lam,{4,5,6});

# Print common eigenspaces
CommonEigenspaces( [ L1, L2, L3, L4, L5, K1, K2, K3, K4 ] );
```

The action of K_1, K_2, K_3 and K_4 are also given by 16×16 integral matrices.

$$K_1 = \begin{pmatrix}
-5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$K_2 = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & 0 \\
0 & 0 & 1 & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 0 \\
1 & -4 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 \\
-1 & 0 & -4 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & -4 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & -4 & 0 \\
2 & -3 & 1 & 3 & -1 & 0 & 2 & 2 & -2 & -2 & 9 & 2 & 1 & -2 & -1 & 0 \\
-1 & 0 & 0 & -2 & 0 & 1 & -4 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & -1 & 0 & -4 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 1 & -1 & 0 & -4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & -1 & 0 & -1 & 0 & -4 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 2 & 2 & 2 & 1 & 2 & -3 & 1 & -2 & -1 & 0 & 3 & -1 & -2 & -2 & 9
\end{pmatrix}$$

$$K_3 = \begin{pmatrix}
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 0 & 3 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
-2 & 0 & 3 & 0 & 3 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\
-3 & 3 & 0 & 3 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
3 & 0 & 3 & 0 & 3 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 2 & 0 & 2 & 3 & 0 & 1 & 0 & 0 & 3 & 0 & 1 & 0 & 0 \\
-1 & 0 & -1 & 0 & 2 & -2 & 0 & 3 & 0 & 1 & 0 & 0 & 3 & 0 & 1 & 0 \\
-3 & 2 & 0 & -1 & 0 & 2 & 1 & 0 & 3 & 0 & 0 & 1 & 0 & 3 & 0 & 0 \\
3 & 0 & 2 & 0 & -1 & -2 & 0 & 1 & 0 & 3 & 0 & 0 & 1 & 0 & 3 & 0 \\
-1 & 5 & -1 & -5 & 1 & 0 & -4 & -2 & 4 & 2 & -6 & 0 & -1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & -3 & 3 & 0 & 2 & 0 & 0 & 3 & 0 & -1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 1 & 3 & 0 & 3 & 0 & 2 & 0 & 0 & 3 & 0 & -1 & 0 \\
2 & 1 & 0 & 1 & 0 & -3 & 2 & 0 & 3 & 0 & 0 & -1 & 0 & 3 & 0 & 0 \\
-2 & 0 & 1 & 0 & 1 & 3 & 0 & 2 & 0 & 3 & 0 & 0 & -1 & 0 & 3 & 0 \\
0 & -4 & -2 & 0 & -1 & -1 & 5 & -1 & 0 & 1 & 0 & -5 & 1 & 4 & 2 & -6
\end{pmatrix}$$

$$K_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & -2 & 0 & 2 & 0 & 0 & 2 & 0 & -2 & 0 & 1 & -2 & 0 & 2 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 2 & 0 & -2 & 0 & 1
\end{pmatrix}$$

Let ϕ_1 and ϕ_2 be the negative and positive roots of the quadratic equation $x^2 - 3x - 9$. That is

$$\phi_1 = \frac{3}{2}(1 - \sqrt{5}) \quad \text{and} \quad \phi_2 = \frac{3}{2}(1 + \sqrt{5}).$$

Then the spectrum of the nine operators $L_1, L_2, L_3, L_4, L_5, K_1, K_2, K_3, K_4$ is summarised in Table A.2.

Spectrum	Basis vector
$-1, 1, 0, -1, 1, 1, \phi_2, \phi_1, 1$	$(3, -3, 0, 3, 0, 0, \phi_1, 0, -\phi_1, 0, 0, \phi_1 + 3, 0, -\phi_1 - 3, 0, 0)$
$-1, 1, 0, -1, 1, 1, \phi_1, \phi_2, 1$	$(3, -3, 0, 3, 0, 0, \phi_2, 0, -\phi_2, 0, 0, \phi_2 + 3, 0, -\phi_2 - 3, 0, 0)$
$-1, 1, 0, 1, -1, 1, \phi_2, \phi_1, 1$	$(3, -2\phi_1 - 3, 0, -2\phi_1 - 3, 0, -6, \phi_1 + 6, 0, \phi_1 + 6, 0, 0, -\phi_2, 0, -\phi_2, 0, 0)$
$-1, 1, 0, 1, -1, 1, \phi_1, \phi_2, 1$	$(3, -\phi_2 - 3, 0, -\phi_2 - 3, 0, -6, \phi_2 + 6, 0, \phi_2 + 6, 0, 0, -\phi_1, 0, -\phi_1, 0, 0)$
$-1, 1, -3, 1, 2, 1, -6, 9, 1$	$(0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0)$
$-1, 1, 3, -1, -2, 1, 6, 3, -5$	$(0, 1, 0 - 1, 0, 0 - 1, 0, 1, 0, 0, 1, 0 - 1, 0, 0)$
$-1, -2, 3, 1, -1, -5, 3, 6, 1$	$(1, 0, 0, 0, 0 - 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
$-1, -2, 3, -1, 1, -5, 3, 6, 1$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
$1, 2, -3, 1, -1, 1, 9, -6, 1$	$(-3, 8, 2, -4, 2, 6, -10, 2, 2, 2, -6, 8, -4, -4, -4, 12)$
$1, 2, -3, -1, 1, 1, 9, -6, 1$	$(3, -4, 2, 4, -2, 0, 2, 2, -2, -2, 6, 0, 0, 0, 0, 0)$
$1, -1, 0, -1, 1, 1, \phi_2, \phi_1, 1$	$(9, -3, 6, 3, -6, 0, \phi_1, -2\phi_1, -\phi_1, 2\phi_1, 0, \phi_1 + 3, -2\phi_1 - 6, -\phi_1 - 3, 2\phi_1 + 6, 0)$
$1, -1, 0, -1, 1, 1, \phi_1, \phi_2, 1$	$(9, -3, 6, 3, -6, 0, \phi_2, -2\phi_2, -\phi_2, 2\phi_2, 0, \phi_2 + 3, -2\phi_2 - 6, -\phi_2 - 3, 2\phi_1 + 6, 0)$
$1, -1, 0, 1, -1, 1, \phi_2, \phi_1, 1$	$(9, -2\phi_1 - 3, 4\phi_1 + 6, -2\phi_1 - 3, 4\phi_1 + 6, -18, \phi_1 + 6, -2\phi_1 - 12, \phi_1 + 6, -2\phi_1 - 12, 0, -\phi_2, 2\phi_2, -\phi_2, 2\phi_2, 0)$
$1, -1, 0, 1, -1, 1, \phi_1, \phi_2, 1$	$(9, -2\phi_2 - 3, 4\phi_2 + 6, -2\phi_2 - 3, 4\phi_2 + 6, -18, \phi_2 + 6, -2\phi_2 - 12, \phi_2 + 6, -2\phi_2 - 12, 0, -\phi_1, 2\phi_1, -\phi_1, 2\phi_1, 0)$
$1, -1, -3, 1, 2, 1, -6, 9, 1$	$(0, 1 - 2, 1 - 2, 0, 1 - 2, 1 - 2, 0, 1 - 2, 1 - 2, 0)$
$1, -1, 3, -1, -2, 1, 6, 2, -5$	$(0, 1 - 2 - 1, 2, 0 - 1, 2, 1 - 2, 0, 1 - 2 - 1, 2, 0)$

Table A.2: The joint eigenvalues and eigenspaces of $L_1, \ldots, L_5, K_1, \ldots, K_4$

Appendix B

Galois and monodromy groups

In Section B.1 we recall the definition and some of the properties of Galois groups for finite maps. In Section B.2 we recall the definition of the equivariant fundamental group and equivariant monodromy.

B.1 The Galois theory of finite maps

Let X and Y be irreducible varieties of equal dimension and $\pi : Y \longrightarrow X$ a dominant morphism of varieties. We consider the induced field extension $K(X) \hookrightarrow K(Y)$. We call the integer $[K(X) : K(Y)]$ the *degree* of π.

Proposition B.1.1. *The degree $[K(X) : K(Y)]$ is finite.*

Proof. By definition $\dim X = \mathrm{trdeg}_{\mathbb{C}} K(X)$ and $\dim Y = \mathrm{trdeg}_{\mathbb{C}} K(Y)$. Since the dimensions are assumed to be equal, $K(X)$ and $K(Y)$ have the same transcendence degree over $\mathbb{C}$ and so $K(X) \hookrightarrow K(Y)$ is an algebraic extension and necessarily has finite degree. □

We will denote the degree of π by d.

Lemma B.1.2. *The generic fibre of π consists of d points. That is, there exists an open dense subset $U \in X$ such that $\pi : \pi^{-1}(U) \longrightarrow U$ has fibres consisting of d points.*

Proof. Since this is a local property we can assume without loss of generality that $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, B$ are affine. We will prove the claim assuming that a finite list of elements of A

$$a_D, s_D^{(1)}, s_D^{(2)}, \ldots, s_D^{(n)}, m_{\det} \in A$$

are invertible. If these elements are not invertible consider the open subset $U = \mathrm{Spec}\, A_\ell$ where A_ℓ is the localisation of A at each of the above elements. The open set U is dense since X is irreducible and we consider the morphism $\pi : \mathrm{Spec}\, B' = \pi^{-1}(U) \longrightarrow U$, since all of the above elements are invertible in A_ℓ, we will see that we can prove the claim for this morphism. Since U is open and dense in X we have proven the claim in general.

Since π is dominant, the corresponding map of rings $A \to B$ is injective, which induces the field extension $K(A) \hookrightarrow K(B)$. Since we are over $\mathbb{C}$ this is a separable algebraic extension of degree d, hence there exists an element $\alpha \in K(A)$ such that $K(B) = K(A)[\alpha]$, where α satisfies a monic polynomial of degree d:

$$P(t) = a_0 + a_1 t + \cdots + a_{d-1} t^{d-1} + t^d \in K(A)[t].$$

Let a_D be the common denominator of the a_i, which we have assumed to be invertible in A, thus $P(t) \in A[t]$.

Suppose B is generated by $x_1, x_2, \dots, x_n$ over A. Since $x_i \in K(A)[\alpha]$ there exist elements $s_j^{(i)} \in K(A)$ such that

$$x_i = s_0^{(i)} + s_1^{(i)} \alpha + \cdots + s_{d-1}^{(i)} \alpha^{d-1}.$$

Let $s_D^{(i)}$ be the common denominator of the $s_j^{(i)}$, which we have assumed to be invertible in A, thus $s_j^{(i)} \in$ and hence $B \subseteq A[\alpha]$.

Now since $\alpha \in K(B) = K(A)[\alpha]$ there exist elements $b, c \in B$ such that $\alpha = b/c$. The elements $1, \alpha, \dots, \alpha^{d-1}$ are linearly independent over A and thus the elements $b^i c^{d-1-i}$ for $i = 0, \dots, d-1$ are as well. Since $b^i c^{d-1-i} \in B \subseteq A[\alpha]$ there exist elements $m_{ij} \in A$ such that $b^i c^{d-1-i} = \sum_{j=0}^{d-1} m_{ij} \alpha^j$. By the linear independence of these sums, the determinant $m_{\det} \in A$ of the matrix (m_{ij}) is nonzero and we assume that it is invertible. By inverting the system of equations we see that $\alpha \in B$ and hence $B = A[\alpha]$.

Let $\Delta \in A$ be the discriminant of P and let $U_\Delta \subseteq X$ be the open set where $\Delta \neq 0$, this is dense since X is irreducible. Let $x \in U_\Delta$, then the fibre $\pi^{-1}(x)$ is the spectrum of the ring

$$k(x) \otimes_A B = k(x) \otimes_A A[\alpha] \cong \frac{k(x)[t]}{(P(t))},$$

where $k(x)$ is the residue field at x. Since the discriminant of $P(t)$ is nonzero at x, it has d distinct roots and hence the fibre is a set of d reduced points. □

Fix a smooth point $p \in X$ in the open set defined by Lemma B.1.2, denote the fibre $\Gamma = \pi^{-1}(p) = \{q_1, q_2, \dots, q_d\}$. We define two subgroups of the symmetric group S_d of these d points.

B.1.1 The Galois group

The definitions and results in this section are from [Har79]. By Lemma B.1.2, the fibre over a generic point $x \in X$ consists of d reduced points, which we denote $y_1, y_2, \dots, y_d$. Let $\alpha \in K(Y)$ and $P \in K(X)[t]$ be as in the proof of Lemma B.1.2, that is, P is a monic polynomial of degree d and α a generator of $K(Y)$ over $K(X)$ such that $P(\alpha) = 0$. Since α generates $K(Y)$ over $K(X)$, if $\alpha(y_i) = \alpha(y_j)$ then every rational function on Y would agree at y_i and y_j.

Let $\mathcal{M}_x$ be the field of germs of meromorphic functions around x and $\mathcal{M}_i$ the field of meromorphic functions around y_i. We have natural inclusions $K(X) \subseteq \mathcal{M}_x$ and $K(Y) \subseteq \mathcal{M}_i$. Since π is locally around y_i an isomorphism of analytic varieties we have isomorphisms $\phi_i : \mathcal{M}_i \longrightarrow \mathcal{M}_x$. Note that the following diagram commutes,

$$\begin{tikzcd} \mathcal{M}_x & \mathcal{M}_i \arrow[l, "\phi_i"'] \\ K(X) \arrow[u, hook] \arrow[r, hook, "\pi^*"] & K(Y). \arrow[u, hook] \end{tikzcd} \tag{B.1.1}$$

Let $K(Y)_i = \phi_i(K(Y))$, and let L be the subfield of $\mathcal{M}_x$ generated by the $K(Y)_i$. By the commutativity of (B.1.1), $K(X) \subset L$. The polynomial $P \in K(X)[t]$ can be thus be thought of as a polynomial in $L[t]$. The images $\alpha_i = \phi_i(\alpha) \in L$ are all distinct (since the value of α is distinct at each y_i) and thus are a complete set of roots for P. The field L is thus the Galois closure, in $\mathcal{M}_x$, of the extension $K(X) \hookrightarrow K(Y)$ and the Galois group $\mathrm{Gal}(L/K(X))$ acts on the set of roots $\{\alpha_i\}$ which we may identify canonically with the set of points in the fibre $\pi^{-1}(x)$.

Definition B.1.3. The image of $\mathrm{Gal}(L/K(X))$ in $S_{\pi^{-1}(x)}$, the group of permutations of the fibre, is called the *Galois group* of π and is denoted $\mathrm{Gal}(\pi)$ or $\mathrm{Gal}(\pi; x)$ if we wish to emphasise the basepoint.

The definition of the Galois group $\mathrm{Gal}(\pi; x)$ depends only on local properties of the morphism π, it is thus a birational invariant of π.

Proposition B.1.4. *The group* $\mathrm{Gal}(\pi) \subset S_d$ *depends only on the rational equivalence class of* π *and not on* π *itself.*

Proof. Let $\pi' : X' \longrightarrow Y'$ be another degree d, dominant morphism and suppose we have birational maps making the following diagram commute,

$$\begin{tikzcd} Y \arrow[r, dashed, "g"] \arrow[d, "\pi"] & Y' \arrow[d, "\pi'"] \\ X \arrow[r, dashed, "f"] & X'. \end{tikzcd}$$

Suppose f is defined on x and g is defined on $y_1, y_2, \ldots, y_d$. The morphisms f and g provide isomorphisms $f^\flat : \mathcal{M}_x \longrightarrow \mathcal{M}_{f(x)}$ and $g_i^\flat : K(Y) \longrightarrow K(Y')$, which restrict to isomorphisms $K(X) \longrightarrow K(X')$ and $K(Y)_i \longrightarrow K(Y')_i$ (and thus also between L and L'). Importantly these isomorphisms send primitive elements to primitive elements and thus after identifying the groups $S_{\pi^{-1}(x)}$ and $S_{\pi'^{-1}(f(x))}$ using g, the Galois groups are equal. $\square$

B.1.2 The monodromy group

Fix a Zariski open subset $U \subseteq X$ containing p and such that $\pi : \pi^{-1}(U) \longrightarrow U$ is an unbranched covering space. By the *unique lifting property* (see [Hat02, Proposition 1.34]) given a loop at p and a choice of point q_i in the fibre, there exists a unique path in Y lifting the original loop, and starting at q_i. This defines an action of the fundamental group $\pi_1(X, p)$ in the fibre $\{q_1, q_2, \ldots, q_d\}$.

Definition B.1.5. The image of the fundamental group in S_d is called the *monodromy group* of π and is denoted $M(U, \pi)$.

B.1.3 Harris' theorem

We have described two actions on the generic fibre of a dominant morphism of varieties of equal dimension. The first, the Galois group, is defined in a natural way and since it depends only on local information around the point p, it depends only on the rational equivalence class of π. The second, the action of the monodromy group, however (a priori) depend on a choice local neighbourhood of p. The following theorem of Harris shows that in fact these two actions coincide and thus the monodromy group does not depend on the choice of U.

Theorem B.1.6 ([Har79, Section 1])**.** *For $\pi : Y \longrightarrow X$ a morphism of irreducible varieties of equal dimension, the Galois group* $\mathrm{Gal}(\pi)$ *and the monodromy group $M(U, \pi)$ (defined for any open set), coincide.*

B.2 Equivariant monodromy

Let G be a topological group. Suppose we have a G-equivariant covering map of locally connected G-spaces

$$S \xrightarrow{f} B.$$

Fix a basepoint $b \in B$. The fundamental group $\pi_1(B, b)$ acts by *monodromy* on the fibre $f^{-1}(b)$. We would like to extend this concept to take into account the G-action.

Let $g \in G$. Choose a path α from b to $g \cdot b$ in B. For any point $p \in f^{-1}(b)$ we can lift α uniquely to a path in S, starting at p and ending at some point $q \in f^{-1}(g \cdot b)$. Since f is G-equivariant, $g^{-1} \cdot q \in f^{-1}(b)$. We could define an action $(\alpha, g) \cdot p = g^{-1} \cdot q$. One could imagine that this action is not obtained by any loop in $\pi_1(B, b)$.

One approach to study this behaviour is to consider the respective quotients and the induced map

$$S/G \xrightarrow{f^G} B/G.$$

However these quotients will often be very pathological topological spaces. However if G acts freely (of factors through a free action) on S and B the map f^G will be a topological covering. We would like our notion of equivariant monodromy to coincide with the usual notion of monodromy for f^G in this situation. We illustrate this with an example.

B.2.1 An example

We consider the map

$$(\mathbb{C}^\times)^2 \xrightarrow{f} \mathbb{C}^\times$$

Given by $f(w,z) = w^2 + z^2$. Let $B = \mathbb{C}^\times$ and restrict the map f to the subspace $S = \{(w,z) \,|\, wz = 0\} \subset (\mathbb{C}^\times)^2$. This a covering of degree 4. We may think of S as two disjoint copies of $\mathbb{C}^\times$ and f as two copies of the map $u \mapsto u^2$.

Choose $1 \in B$ as our basepoint. The fundamental group $\pi_1(B,1) \cong \mathbb{Z}$ is generated by the path $\gamma : r \mapsto e^{2r\pi i}$. The fibre over 1 is

$$f^{-1}(1) = \{(1,0), (-1,0), (0,1), (0,-1)\}\,.$$

The unique lift of the path γ to a path in S starting at $(1,0)$ is $\tilde{\gamma} : r \mapsto (e^{r\pi i})$. So $\gamma \cdot (1,0) = (-1,0)$. Similarly $\gamma \cdot (0,1) = (0,-1)$ and γ^2 acts as the identity.

$$\begin{tikzcd} f^{-1}(1): & (1,0) \arrow[r, bend left, "\gamma"] & (-1,0) \arrow[l, bend left, "\gamma"] & (0,1) \arrow[r, bend left, "\gamma"] & (0,-1) \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

The cyclic group of order two $C_2 = \{1, \zeta\}$ acts on B by $\zeta \cdot z = -z$ and on $(\mathbb{C}^\times)^2$ by $\zeta \cdot (w,z) = (-iz, iw)$, S is stable under this action. The map f is C_2-equivariant.

Consider the path $\alpha : r \mapsto e^{r\pi i}$ in B. The path α starts at 1 and ends at $-1 = \zeta \cdot 1$. The unique lift of α to a path in S starting at $(1,0)$ is the path $\tilde{\alpha} : r \mapsto (e^{\frac{1}{2}r\pi i}, 0)$. Now $\tilde{\alpha}(1) = (i,0) = \zeta \cdot (0,-1)$. Following the above discussion we could say that we have C_2-equivariant monodromy, $(\alpha,\zeta) \cdot (1,0) = (0,-1)$. Similarly we can calculate the action of (α,ζ) on the other points in the fibre:

$$\begin{tikzcd} f^{-1}(1): & (1,0) \arrow[r, bend left] \arrow[rrr, bend right] & (-1,0) \arrow[l, bend left, "\gamma"] \arrow[r, bend right] & (0,1) \arrow[ll, bend right] \arrow[r, bend left] & (0,-1) \arrow[l, bend left] \arrow[ll, bend right, "{(\alpha,\zeta)}"'] \end{tikzcd}$$

We can see that we realise an action of a cyclic group of order 4 on $f^{-1}(1)$ in this way. That is, we have produced extra monodromy using the group action.

To see this coincides with the picture when we take quotients, first note that C_2 acts freely on both $B/C_2 \cong \mathbb{C}^\times$ by the map $C_2 \cdot z \mapsto z^2$ and $S/C_2 \cong \mathbb{C}^\times$ by the map $C_2 \cdot (w, z) \mapsto w - iz$. The induced map is

$$\begin{array}{ccccccc} \mathbb{C}^\times & \xrightarrow{\sim} & S/C_2 & \xrightarrow{f^{C_2}} & B/C_2 & \xrightarrow{\sim} & \mathbb{C}^\times \\ u & \longmapsto & C_2 \cdot (u, 0) & \longmapsto & C_2 \cdot u^2 & \longmapsto & u^4. \end{array}$$

So we see that f^{C_2} is a covering of degree four. If we choose our basepoint to be $C_2 \cdot 1 \in B/C_2$ the fibre is

$$f^{-1}(C_2 \cdot 1) = \{C_2 \cdot (1, 0), C_2 \cdot (-1, 0), C_2 \cdot (0, 1), C_2 \cdot (0, -1)\},$$

and the action of the generator of $\pi^{-1}(B/C_2, C_2 \cdot 1)$ is exactly as described in the above diagram.

B.2.2 The equivariant fundamental group

The above example shows that our informal notion of an equivariant path or monodromy is a good candidate. We will formalise this in a definition. We follow the definition originally given by Rhodes in [Rho66]. It is possible to formulate the definitions in many languages, some of which apply in much broader generality (i.e the language of orbifolds or stacks).

Definition B.2.1. Let X be a topological space and G a discrete group acting on X. Let $b \in X$ be a basepoint. The *equivariant fundamental group* $\pi_1^G(X, b)$ has elements (α, g) where $g \in G$ and α is a (homotopy class of a) path from b to $g \cdot b$. The group structure is defined by

$$(\alpha, g) \cdot (\beta, h) = (\alpha \cdot g(\beta), gh).$$

Here we use the usual composition of paths and $g(\beta)$ to denote the g-translate of the path β. The elements (α, g) are called *G-equivariant paths.*

From the definition we can see the set of elements of the form $(\alpha, 1)$ forms a subgroup of $\pi_1^G(X, b)$ isomorphic to $\pi_1(X, b)$. In fact if we consider the projection $(\alpha, g) \mapsto g$ we obtain a surjective group homomorphism from $\pi_1^G(X, b)$ to G. The kernel of this homomorphism is the above described subgroup so we have an exact sequence

$$1 \longrightarrow \pi_1(X, b) \longrightarrow \pi_1^G(X, b) \longrightarrow G \longrightarrow 1.$$

Proposition B.2.2 ([Rho66, Theorem 1]). *If $b, c \in X$ are path connected basepoints, and δ a homotopy class of paths from b to c the induced map*

$$\pi_1^G(X, b) \to \pi_1^G(X, c)$$

$$(\alpha, g) \mapsto (\delta^{-1} \cdot \alpha \cdot g(\delta), g),$$

is an isomorphism.

Let X and Y be G spaces and f and G-equivariant map from X to Y. As is the case with the usual fundamental group, we have an induced homomorphism

$$f_* : \pi_1^G(X, b) \longrightarrow \pi_1^G(Y, f(b)),$$

given by sending (α, g) to $(f_*\alpha, g)$. Because of the G-equivairence of f this is easily checked to be a homomorphism.

If X and Y are homotopic spaces we say that are G-homotopic if we can find G-equivariant maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.

Proposition B.2.3 ([Rho66, Theorem 3])**.** *Let X and Y be G-spaces which are G-homotopic via the G-equivariant map $f : X \longrightarrow Y$. Then the induced homomorphism*

$$f_* : \pi_1^G(X, b) \longrightarrow \pi_1^G(Y, f(b)),$$

is an isomorphism. That is π_1^G in a G-homotopy invariant.

As an almost immediate consequence of the definition we have the following Lemma.

Lemma B.2.4. *Suppose H is a subgroup of G which acts on a topological space X. Then the equivariant fundamental group $\pi_1^H(X, b)$ is the kernel of the composition*

$$\pi_1^G(X, b) \longrightarrow G \longrightarrow G/H,$$

of the projection and canonical quotient map.

Definition B.2.5. If $f : S \longrightarrow B$ is a G-equivariant topological covering we define an action of the group $\pi_1^G(X, b)$ on the fibre $f^{-1}(b)$. If $p \in f^{-1}(b)$ and $(\alpha, g) \in \pi_1^G(X, b)$ then denote by $\tilde{\alpha}$ the unique lift of α to S such that $\tilde{\alpha}(0) = p$. Since f is G-equivariant $\tilde{\alpha}(1) \in f^{-1}(g \cdot b)$. Define $(\alpha, g) \cdot p = g^{-1} \cdot \tilde{\alpha}(1)$. We call this the *$G$-equivariant monodromy action* of $\pi_1^G(B, b)$ on $f^{-1}(b)$.

Appendix C

Finite and flat maps

In this appendix we recall a criterion for checking flatness of finite maps and prove a result allowing us to check isomorphism of finite and flat families fibrewise at closed points. The results in this chapter are almost certainly well known, however due to lack of a precise reference we include proofs.

Let S be a commutative ring. Recall that the *Jacobson radical* $J(S)$ of S is the intersection of all maximal ideals of S. A ring where $J(S) = 0$ is called a *Jacobson ring*. The ring S is *finite type* over $\mathbb{C}$ if it is a finitely generated $\mathbb{C}$-algebra. The following is a result we will use many times in this appendix.

Lemma C.0.1. *If S is a finite type algebra over $\mathbb{C}$, then* $\mathrm{nil}(S) = J(S)$.

Proof. Since S is a finitely generated commutative algebra, there is a surjective homomorphism of algebras

$$\pi : A = \mathbb{C}[x_1, x_2, \dots, x_n] \longrightarrow S,$$

for some n. Let $I = \ker(\pi)$. By Hilbert's Nullstellensatz,

$$\sqrt{I} = \left\{ p \in A \,\middle|\, p^k \in I \text{ for } k \gg 0 \right\} = \bigcap_{\mathfrak{m} \supset I} \mathfrak{m},$$

where the intersection ranges over maximal ideals of A containing I. By definition, $\pi(\sqrt{I}) = \mathrm{nil}(S)$. The maximal ideals of $S \cong A/I$ are exactly the reduction modulo I of the maximal ideals of A containing I. Thus

$$J(S) = \pi(\bigcap_{\mathfrak{m} \supset I} \mathfrak{m}) = \mathrm{nil}(S).$$

□

C.1 Constant dimensional fibres

We present a criterion for checking flatness of a finitely generated module M over a commutative Noetherian ring S. Recall that a finitely generated and projective module is flat. Define the function

$$\varphi(\mathfrak{p}) = \dim_{S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}} \left(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}\right). \tag{C.1.1}$$

Recall that a function $f : X \longrightarrow \mathbb{R}$ is *upper semicontinuous* on a topological space X if the set

$$\{x \in X \mid f(x) < a\}$$

is open for any $a \in \mathbb{R}$. The following result is well known (see [Har77, Exercise II.5.8] or [Eis95, Exersise 20.13])

Proposition C.1.1. *If M is finitely generated then φ is upper semicontinuous. If S is reduced and φ is constant then M is a projective S-module.*

C.1.1 Checking flatness at closed points

The aim of this section of to show, in certain situations, that we can apply Proposition C.1.1 by checking that (C.1.1) is constant on the locus of closed points.

Lemma C.1.2. *Suppose R is a Jacobson ring, then any nonempty, Zariski-open set $U \subset \operatorname{Spec} S$ contains a maximal ideal.*

Proof. Suppose for contradiction U contains no maximal ideals. By the definition of the Zariski topology $\operatorname{Spec} S - U = V(I)$ for some ideal $I \subset S$. Since U contains no maximal ideal, $V(I)$ must contain every maximal ideal. Thus $I \subset J(S)$. By assumption this means $I = 0$ so $V(I) = \operatorname{Spec} S$ and $U = \emptyset$. $\square$

Lemma C.1.3. *Let $\eta : \operatorname{Spec} S \longrightarrow \mathbb{R}$ be an upper semicontinuous function with the property that $\eta(\mathfrak{p}) \leq \eta(\mathfrak{q})$ if $\mathfrak{p} \subset \mathfrak{q}$ for every $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} S$. If S is a Jacobson ring and if η is constant on the set of maximal ideals then η is constant on $\operatorname{Spec} S$.*

Proof. Let $n = \eta(\mathfrak{m})$, for any maximal ideal $\mathfrak{m} \subset S$ (since η is constant on maximal ideal by assumption, the choice of $\mathfrak{m}$ doesn't matter). Fix a prime ideal $\mathfrak{p} \in \operatorname{Spec} S$. Since every prime ideal is contained in a maximal ideal, $\eta(\mathfrak{p}) \leq n$. Consider the set

$$U = \{\mathfrak{q} \in \operatorname{Spec} S \mid \eta(\mathfrak{q}) < n\}.$$

By the definition of upper semicontinuity U is open, and furthermore does not contain any maximal ideals. Therefore, by Lemma C.1.2 $U = \emptyset$, in particular $\mathfrak{p} \notin U$ and thus we have $\eta(\mathfrak{p}) \geq \eta(\mathfrak{m}) = n$. $\square$

Lemma C.1.4. *Let $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of S, then $\varphi(\mathfrak{p}) \leq \varphi(\mathfrak{q})$.*

Proof. Let $n = \varphi(\mathfrak{q})$ and choose a basis $x_1, x_2, \ldots, x_n$ of $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$ over the field $S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$. By Nakayama's Lemma (see [Eis95, Corollary 4.8]) we can lift this to a minimal set of generators $y_1, y_2, \ldots, y_n$ of $M_\mathfrak{q}$ over the ring $S_\mathfrak{q}$. Since $\mathfrak{p} \subset \mathfrak{q}$, $y_1, y_2, \ldots, y_n$ also generate $M_\mathfrak{p}$ over $S_\mathfrak{p}$ (though the set may no longer be minimal) and hence their projections span $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ over the field $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$. □

Proposition C.1.5. *Let S be an algebra of finite type over $\mathbb{C}$ and let M be a finitely generated module. If φ is constant on the set of maximal ideals, then M is projective.*

Proof. By Proposition C.1.1, φ is upper semicontinuous and to prove M is projective we must check that φ is constant on the entirety of $\operatorname{Spec} S$. Since S is of finite type, by Lemma C.0.1, $\operatorname{nil}(S) = J(S)$, and since S is reduced, $J(S) = 0$. Lemma C.1.4 says that $\varphi(\mathfrak{p}) \le \varphi(\mathfrak{q})$ for any two prime ideals $\mathfrak{p} \subset \mathfrak{q}$ and thus we can apply Lemma C.1.3 which shows that φ is constant on $\operatorname{Spec} S$. □

C.2 Fibrewise isomorphisms

In this section we prove a result which allows us to check whether a given morphism between affine schemes is an isomorphism by checking if it is an isomorphism on fibres over closed points.

C.2.1 Preliminary results

The radical $\operatorname{rad}(M)$ of an S-module M is the intersection of all maximal submodules.

Lemma C.2.1. *For any S-module M we have $\operatorname{rad}(M) = \bigcap_\mathfrak{m} \mathfrak{m}M$, where $\mathfrak{m}$ ranges over the set of maximal ideals of S.*

Proof. See [AF92, Exercise 15.5]. □

Lemma C.2.2. *Let P be a projective S-module. Then $\operatorname{rad}(P) = J(S)P$.*

Proof. See for example [AF92, Proposition 17.10]. □

Lemma C.2.3. *Let S be a reduced, finitely generated algebra over C and P a projective S-module. For any maximal ideal $\mathfrak{m} \subset S$, let $\pi_\mathfrak{m} : P \longrightarrow P/\mathfrak{m}P$ be the canonical projection. If $a \in P$ has the property that $\pi_\mathfrak{m}(a) = 0$, for every maximal ideal, then $a = 0$.*

Proof. Since $\pi_\mathfrak{m}(a) = 0$ for all maximal ideals, $a \in \bigcap_\mathfrak{m} \mathfrak{m}P$. By Lemma C.2.1 $a \in \operatorname{rad}(P)$. But by Lemma C.2.2, $\operatorname{rad}(P) = J(S)P$. Since S is a finitely generated algebra, Lemma C.0.1 says that $J(S) = \operatorname{nil}(S)$ and since S is in addition reduced $J(S) = \operatorname{nil}(S) = 0$. In particular $\operatorname{rad}(P) = 0$. □

C.2.2 The main result

Let L be a complex affine algebraic variety and let $\varphi : X \longrightarrow Y$ be an isomorphism of affine schemes over L. Let $\tilde{X}$ and $\tilde{Y}$ be affine subschemes of X and Y respectively, flat and finite over L.

Proposition C.2.4. *Suppose, for every close point $x \in L$, φ restricts to an isomorphism of scheme theoretic fibres $\varphi_x : \tilde{X}_x \longrightarrow \tilde{Y}_x$. Then the restriction of φ to $\tilde{X}$ induces an isomorphism*

$$\tilde{\varphi} : \tilde{X} \longrightarrow \tilde{Y}.$$

Proof. We restate the problem algebraically. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and let $L = \operatorname{Spec} S$. Let I be the ideal of A cutting out $\tilde{X}$ and J the ideal of B cutting out $\tilde{Y}$. We are given an isomorphism $\varphi^{\#} : B \longrightarrow A$ and we would like to show $\varphi^{\#}(J) = I$. The condition that φ_x is an isomorphism for every closed point $x \in L$, translates to the fact that for every maximal ideal $\mathfrak{m} \subset S$,

$$\varphi^{\#}(\mathfrak{m}B + J) = \mathfrak{m}A + I.$$

But $\varphi^{\#}(\mathfrak{m}B + J) = \varphi^{\#}(\mathfrak{m}B) + \varphi^{\#}(J)$ and since $\varphi^{\#}$ is an isomorphism of S-algebras, $\varphi^{\#}(\mathfrak{m}B) = \mathfrak{m}A$. Thus we know, that for any maximal ideal $\mathfrak{m} \subset S$,

$$\mathfrak{m}A + \varphi^{\#}(J) = \mathfrak{m}A + I. \tag{C.2.1}$$

Now $P = A/I$ is finitely generated and flat over the Noetherian ring S and is hence projective. Let $\pi_{\mathfrak{m}} : P \longrightarrow P/\mathfrak{m}P$ be the canonical projection. Let $a \in \varphi^{\#}(J)$, by (C.2.1), $\pi_{\mathfrak{m}}(a) = 0$. Since this is true for all maximal ideals, Lemma C.2.3 implies that $a \in I$. Thus $\varphi^{\#}(J) \subset I$. Applying the same argument to the projective module $A/\varphi^{\#}(J)$ we see that $I \subset \varphi^{\#}(J)$. $\square$

Bibliography

[AF92] F. W. Anderson and K. R. Fuller. *Rings and categories of modules.* Second. Vol. 13. Graduate Texts in Mathematics. New York, NY: Springer-Verlag, New York, 1992.

[AFV11] L. Aguirre, G. Felder, and A. P. Veselov. "Gaudin subalgebras and stable rational curves". In: *Compositio Mathematica* 147.5 (2011), pp. 1463–1478.

[BCP97] W. Bosma, J. Cannon, and C. Playoust. "The Magma algebra system. I. The user language". In: *Journal of Symbolic Computation* 24.3-4 (1997), pp. 235–265.

[BK95] A. D. Berenstein and A. N. Kirillov. "Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux". In: *Rossi skaya Akademiya Nauk. Algebra i Analiz* 7.1 (1995), pp. 92–152.

[BR13] C. Bonnafé and R. Rouquier. "Cellules de Calogero-Moser". In: *arXiv.org* (Feb. 2013). arXiv: `1302.2720 [math.RT]`.

[Che94] I. Cherednik. "Integration of quantum many-body problems by affine Knizhnik-Zamolodchikov equations". In: *Advances in Mathematics* 106.1 (1994), pp. 65–95.

[Dev99] S. L. Devadoss. "Tessellations of moduli spaces and the mosaic operad". In: *Homotopy invariant algebraic structures (Baltimore, MD, 1998).* Providence, Rhode Island: Amer. Math. Soc., Providence, RI, 1999, pp. 91–114.

[DJS03] M. Davis, T. Januszkiewicz, and R. Scott. "Fundamental groups of blow-ups". In: *Advances in Mathematics* 177.1 (2003), pp. 115–179.

[EG02] P. I. Etingof and V. Ginzburg. "Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism". In: *Inventiones mathematicae* 147.2 (2002), pp. 243–348.

[EH83] D. Eisenbud and J. Harris. "Divisors on general curves and cuspidal rational curves". In: *Inventiones mathematicae* 74.3 (1983), pp. 371–418.

[Eis95] D. Eisenbud. *Commutative algebra.* Vol. 150. Graduate Texts in Mathematics. New York: Springer-Verlag, 1995.

[FFR94] B. Feigin, E. Frenkel, and N. Y. Reshetikhin. “Gaudin model, Bethe ansatz and critical level”. In: *Communications in Mathematical Physics* 166.1 (1994), pp. 27–62.

[FFT10] B. Feigin, E. Frenkel, and V. Toledano Laredo. “Gaudin models with irregular singularities”. In: *Advances in Mathematics* 223.3 (2010), pp. 873–948.

[Ful97] W. Fulton. *Young tableaux*. Vol. 35. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997.

[Gau76] M. Gaudin. “Diagonalisation d’une classe d’Hamiltoniens de spin”. In: *Le Journal de Physique* 37.10 (1976), pp. 1089–1098.

[Gau83] M. Gaudin. *La fonction d’onde de Bethe*. Collection du Commissariat à l’Énergie Atomique: Série Scientifique. [Collection of the Atomic Energy Commission: Science Series]. Masson, Paris, 1983.

[GH94] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley Classics Library. Hoboken, NJ, USA: John Wiley & Sons, Inc., New York, 1994.

[GW10] U. Görtz and T. Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Wiesbaden: Vieweg + Teubner, Wiesbaden, 2010.

[Hai92] M. D. Haiman. “Dual equivalence with applications, including a conjecture of Proctor”. In: *Discrete Mathematics* 99.1-3 (1992), pp. 79–113.

[Har77] R. Hartshorne. *Algebraic geometry*. New York: Springer-Verlag, 1977.

[Har79] J. Harris. “Galois groups of enumerative problems”. In: *Duke Mathematical Journal* 46.4 (1979), pp. 685–724.

[Hat02] A. Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.

[HK02] J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases*. Vol. 42. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2002.

[HK06] A. Henriques and J. Kamnitzer. “Crystals and coboundary categories”. In: *Duke Mathematical Journal* 132.2 (2006), pp. 191–216.

[JK81] G. James and A. Kerber. *The representation theory of the symmetric group*. Vol. 16. Encyclopedia of Mathematics and its Applications. Addison-Wesley Publishing Co., Reading, Mass., 1981.

[Juc74] A. A. A. Jucys. “Symmetric polynomials and the center of the symmetric group ring”. In: *Reports on Mathematical Physics* 5.1 (1974), pp. 107–112.

[Kap93] M. M. Kapranov. "The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation". In: *Journal of Pure and Applied Algebra* 85.2 (1993), pp. 119–142.

[Kas90] M. Kashiwara. "Crystalizing the q-analogue of universal enveloping algebras". In: *Communications in Mathematical Physics* 133.2 (1990), pp. 249–260.

[Kas91] M. Kashiwara. "On crystal bases of the Q-analogue of universal enveloping algebras". In: *Duke Mathematical Journal* 63.2 (1991), pp. 465–516.

[Kle05] A. Kleshchev. *Linear and projective representations of symmetric groups.* Vol. 163. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 2005.

[KN94] M. Kashiwara and T. Nakashima. "Crystal graphs for representations of the q-analogue of classical Lie algebras". In: *Journal of Algebra* 165.2 (1994), pp. 295–345.

[Knu70] D. E. Knuth. "Permutations, matrices, and generalized Young tableaux". In: *Pacific Journal of Mathematics* 34 (1970), pp. 709–727.

[Knu83] F. F. Knudsen. "The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$". In: *Mathematica Scandinavica* 52.2 (1983), pp. 161–199.

[Koh83] T. Kohno. "On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces". In: *Nagoya Mathematical Journal* 92.2 (1983), pp. 21–37.

[KV07] J. Kock and I. Vainsencher. *An invitation to quantum cohomology.* Vol. 249. Progress in Mathematics. Boston, MA: Birkhäuser Boston Inc., 2007.

[Lev15] J. Levinson. "One-dimensional Schubert problems with respect to osculating flags". In: *arXiv.org* (Apr. 2015). arXiv: `1504.06542v1 [math.AG]`.

[Lot02] M. Lothaire. *Algebraic combinatorics on words.* Vol. 90. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, Cambridge, 2002.

[Mar10] A. Marcus. "A Bethe Ansatz for Symmetric Groups". In: *arXiv.org* (Mar. 2010). arXiv: `1003.0490v1 [math.RT]`.

[Mat92] A. Matsuo. "Integrable connections related to zonal spherical functions". In: *Inventiones mathematicae* 110.1 (1992), pp. 95–121.

[MTV06] E. Mukhin, V. Tarasov, and A. N. Varchenko. "Bethe eigenvectors of higher transfer matrices". In: *Journal of Statistical Mechanics: Theory and Experiment* 2006.8 (2006), P08002–44.

[MTV09a] E. Mukhin, V. Tarasov, and A. N. Varchenko. “Schubert calculus and representations of the general linear group”. In: *Journal of the American Mathematical Society* 22.4 (2009), pp. 909–940.

[MTV09b] E. Mukhin, V. Tarasov, and A. N. Varchenko. “The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz”. In: *Annals of Mathematics. Second Series* 170.2 (2009), pp. 863–881.

[MTV10] E. Mukhin, V. Tarasov, and A. N. Varchenko. “Gaudin Hamiltonians generate the Bethe algebra of a tensor power of the vector representation of $\mathfrak{gl}_N$”. In: *Rossi skaya Akademiya Nauk. Algebra i Analiz* 22.3 (2010), pp. 177–190.

[MTV12] E. Mukhin, V. Tarasov, and A. N. Varchenko. “Three sides of the geometric Langlands correspondence for gl_N Gaudin model and Bethe vector averaging maps”. In: *Arrangements of hyperplanes—Sapporo 2009.* Math. Soc. Japan, Tokyo, 2012, pp. 475–511.

[MTV13] E. Mukhin, V. Tarasov, and A. N. Varchenko. “Bethe subalgebras of the group algebra of the symmetric group”. In: *Transformation Groups* 18.3 (2013), pp. 767–801.

[MTV14] E. Mukhin, V. Tarasov, and A. N. Varchenko. “Bethe algebra of Gaudin model, Calogero-Moser space, and Cherednik algebra”. In: *International Mathematics Research Notices. IMRN* 5 (2014), pp. 1174–1204.

[Mur83] G. E. Murphy. “The idempotents of the symmetric group and Nakayama’s conjecture”. In: *Journal of Algebra* 81.1 (1983), pp. 258–265.

[MV04] E. Mukhin and A. N. Varchenko. “Critical points of master functions and flag varieties”. In: *Communications in Contemporary Mathematics* 6.1 (2004), pp. 111–163.

[OV96] A. Okounkov and A. Vershik. “A new approach to representation theory of symmetric groups”. In: *Selecta Mathematica. New Series* 2.4 (1996), pp. 581–605.

[Rho66] F. Rhodes. “On the fundamental group of a transformation group”. In: *Proceedings of the London Mathematical Society. Third Series* 16.1 (1966), pp. 635–650.

[Rob38] G. d. B. Robinson. “On the Representations of the Symmetric Group”. In: *American Journal of Mathematics* 60.3 (1938), pp. 745–760.

[RV95] N. Y. Reshetikhin and A. N. Varchenko. “Quasiclassical asymptotics of solutions to the KZ equations”. In: *Geometry, topology, & physics.* Int. Press, Cambridge, MA, 1995, pp. 293–322.

[Ryb14] L. Rybnikov. "Cactus group and monodromy of Bethe vectors". In: *arXiv.org* (Aug. 2014). arXiv: `1409.0131v2 [math.QA]`.

[Sch61] C. Schensted. "Longest increasing and decreasing subsequences". In: *Canadian Journal of Mathematics. Journal Canadien de Mathématiques* 13 (1961), pp. 179–191.

[Sch77] M.-P. Schützenberger. "La correspondance de Robinson". In: *Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976)*. Springer, Berlin, 1977, 59–113. Lecture Notes in Math., Vol. 579.

[Spe14] D. E. Speyer. "Schubert problems with respect to osculating flags of stable rational curves". In: *Algebraic Geometry* 1.1 (2014), pp. 14–45.

[Sta99] R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1999.

[Tal06] D. V. Talalaev. "The quantum Gaudin system". In: *Rossi skaya Akademiya Nauk. Funktsional ny Analiz i ego Prilozheniya* 40.1 (2006), pp. 86–91.

[Var95] A. N. Varchenko. "Asymptotic solutions to the Knizhnik-Zamolodchikov equation and crystal base". In: *Communications in Mathematical Physics* 171.1 (1995), pp. 99–137.

[Whi15] N. White. "The monodromy of real Bethe vectors for the Gaudin model". In: *arXiv.org* (Nov. 2015). arXiv: `1511.04740v1 [math.RT]`.

Index of Notation

Index